Piecewise Probability Density Theory

by

Ph. D. & Dr. Sc. Lev Gelimson

Academic Institute for Creating Fundamental Sciences (Munich, Germany)

The "Collegium" All World Academy of Sciences

Munich (Germany)

2nd Edition (2013)

1st Edition (2012)

Abstract

A general piecewise (including pointwise) probability density with short (reduced) notation and its hierarchical particular cases are considered. The explicit closed-form normalization, expectation, and variance formulas along with the median and mode formulas and algorithms for a general one-dimensional piecewise linear probability density are obtained. They are also applied to a general polygonal, or one-dimensional piecewise linear continuous, probability density and, in particular, to a tetragonal probability density. The known formulas for the last density and for a triangular probability density as a further particular case are used to test the developed formulas and algorithms.

2020 Mathematics Subject Classification: primary 60E05; secondary 62E15, 62E17.

Keywords: piecewise probability density, pointwise probability density, piecewise linear probability density, piecewise linear continuous probability density, polygonal probability distribution, polygonal probability density, tetragonal probability density, generalized trapezoidal probability distribution, triangular probability distribution, triangular probability density, mean, median, mode, variance, short notation, abbreviated notation.

Introduction

Both particular and some more general mostly continuous (continual without discontinuity points and jumps) piecewise linear probability densities which can also be multidimensional are well known [Cramér]. For a triangular probability distribution (or, more exactly, density to explicitly indicate namely a probability density rather than a cumulative, or integral, probability distribution function), some basic formulas are also well known [Kotz & van Dorp, Wikipedia Triangular distribution]. [Kotz & van Dorp, van Dorp & Kotz] introduced trapezoidal distributions also generalized including partially nonlinear. [Kim] took into account truncated triangular and trapezoidal distributions also with discontinuity points and jumps. [Karlis & Xekalaki] considered such polygonal distributions only which are expressible via positively weighted sums of independent triangular probability distributions on the unit segment [0, 1]. The mean of such a known polygonal distribution is placed in the middle third part of this unit segment only.
The present work is devoted to presenting a general piecewise (including pointwise) probability density (with short, or abbreviated, notation) and its hierarchical particular cases. The same holds for analytically solving some fundamental problems for piecewise linear probability densities which may be discontinuous and are namely directly introduced, which ensures most possible generality. They are very simple, natural, and typical and can provide adequately modeling via efficiently approximating practically arbitrary nonlinear probability densities with any desired and/or required precision. General polygonal, or one-dimensional piecewise linear continuous, probability densities are also very important extensions of tetragonal and triangular probability densities. It is very natural to verify analytical methods of solving problems for general piecewise linear probability densities via using some well-known basic formulas for a generalized trapezoidal probability distribution and for a triangular probability distribution. Geometrical approach can be also used to additionally verify analytical methods. If there are too many possible cases, which is typical for any piecewise problems, then apply algorithmic approach rather than search for explicit analytical closed-form solutions. The problems of the existence and uniqueness of the mean, median, and mode values for a general one-dimensional piecewise linear probability density are often nontrivial and can be of great importance for practice. It is very useful to provide clear mathematical (probabilistic and statistical) sense of methods and results. Setting and solving many typical urgent problems is the only criterion of creating, developing, and estimating any new useful theory. There are such problems not only in probability theory and mathematical statistics, but also in physics, engineering, chemistry, biology, medicine, geology, astronomy, meteorology, agriculture, politics, management, economics, finance, psychology, etc.

1. General Theoretical Foundations

1.1. Essential Support of a Possibly Non-Measurable Set

To simplify probability distribution analysis and its visualization, reducing a distribution support with possibly nonzero values via excluding its zero-measure and hence zero-measure-dense parts without any influence on integration is very useful if possible.

Nota bene: It is necessary to avoid the following terminology confusion. The mathematical concept of a density is commonly used with two fully different senses:

1) as a purely qualitative density to note the fact of total closeness, nearness, and hence approximability. Namely, a subset A of a set B in a topological space is dense everywhere in the set B or in its subset B' if and only if for any element b of the set B or in its subset B' , respectively, and for any open neighborhood of this element b , this neighborhood contains at least one element a of the subset A . Such a density can be named, e.g., a presence density or an availability density;

2) as a quantitative density, namely as the result of dividing one quantity by another appropriate quantity. This result may be both absolute with physical units (e.g. the average density of a body whose mass and volume are known) and relative without physical units (e.g. a quote of the maximal possible value). Such a density can be named, e.g., a measure density.

Let A be a subset of a topological space S with measure μ giving any open subset U of S a positive number $\mu(U)$. Let

$$\mu^*(A) = \inf\{\mu(U) \mid A \subseteq U \subseteq S\},$$

be the (clearly non-negative) outer measure of A . Let x be a point of S , as well as V containing x be any open subset of S . Then define and determine the average outer density

$$\delta^*(A , V , x) = \mu^*(A \cap V)/\mu(V)$$

of set A at point x with respect to open neighborhood V of x . Let additionally a topological space S be a metric space with a non-negative distance d(x , y) and with a finite of infinite diameter

$$D(B) = \sup\{d(x , y) \mid x \in B , y \in B\}$$

of any subset B of S . Then define and determine the local outer density

$$\delta^*(A , x) = \limsup_{D(V)\to 0+} \delta^*(A , V , x) = \limsup_{D(V)\to 0+} \mu^*(A \cap V)/\mu(V)$$

of set A at point x .

In particular, let μ be the Lebesgue measure λ , as well as a metric space S be the Euclidean n-dimensional space R^n .

For a Lebesgue measurable subset A of the Euclidean n-dimensional space R^n with the Lebesgue measure λ , both the average density

$$\delta_\varepsilon(x) = \lambda(A \cap B_\varepsilon(x))/\lambda(B_\varepsilon(x))$$

of A in a ε-neighborhood of a point x in R^n with the closed ball $B_\varepsilon(x)$ of radius ε centered at a point x and the local density

$$\delta(x) = \lim_{\varepsilon\to 0+} \delta_\varepsilon(x)$$

at a point x along with Lebesgue's density theorem (the density exists and is equal to 1 almost everywhere on A , as well as to 0 almost everywhere on $R^n \setminus A$) are well known [Lebesgue, Halmos, Natanson, Encyclopaedia of Mathematics]. This theorem clearly shows that Lebesgue non-measurable sets are more than typical rather than exceptional and artificial only.

Theorem. In a topological space with measure, a nonzero measure density implies a presence density.

Proof by contradiction. Let b be such an element of the set B or of its subset B' at which a presence density of a subset A of a set B in the set B or in its subset B' does not hold. Then there exists such an open neighborhood of this element b that this neighborhood contains no elements of the subset A

. Morover, this also holds for any smaller open neighborhood of this element b . Hence the set of all the elements of the subset A in any of these open neighborhoods is empty ($\varnothing$). Then the measure density of the subset A of a set B in the set B or in its subset B' is namely zero, which contradicts the given condition of the theorem.

Notata bene:

1. It is not necessary that a measure density exists as the common value of the least measure density and of the greatest measure density. Even in the non-measurability case, it is sufficient that the greatest measure density is nonzero, i.e. strictly positive.
2. In a topological space with measure, a presence density can hold by a zero measure density, and not only locally, but also totally. For example, the rational numbers are both presence-dense and zero-measure-dense namely everywhere in the real numbers.

It is also well known [Lebesgue, Halmos, Natanson, Kolmogorov & Fomin, Encyclopaedia of Mathematics] that for any possibly Lebesgue non-measurable subset A of an abstract space with the Lebesgue measure λ :

1) the outer measure $\lambda^*(A)$ as the greatest lower bound (infimum) of all the (clearly Lebesgue measurable) open covers of A always exists;

2) the inner measure $\lambda_*(A)$ as the least upper bound (supremum) of all the (clearly Lebesgue measurable) closed subsets of A always exists;

3) inequality

$$0 \leq \lambda_*(A) \leq \lambda^*(A)$$

always holds;

4) a subset A of an abstract space with the Lebesgue measure λ is called Lebesgue measurable if and only if its inner and outer measures coincide;

5) if

$$\lambda^* = 0,$$

then

$$\lambda_* = 0 = \lambda^*$$

and such a subset A of an abstract space is Lebesgue measurable and namely a zero-measure set;

6) on any zero-measure set, the Lebesgue integral of any function (whose even all values may be infinite) vanishes.

In, let us now consider any possibly Lebesgue non-measurable subset A with namely strictly positive outer measure

$$\lambda^*(A) > 0.$$

Definition. Essential support

$${}^{E}S(A) = {}^{E}\mathrm{supp}(A)$$

of a possibly non-measurable set A as such a subset of an abstract space S with measure μ that has namely strictly positive outer measure

$$\mu^*(A) > 0$$

is the set A without all its zero-measure-dense points so that

$${}^{E}S(A) = {}^{E}\mathrm{supp}(A) = A \setminus \{x \in A \mid d(x) = 0\}$$

with the same strictly positive outer measure

$$\mu^*({}^{E}S(A)) = \mu^*(A) > 0$$

due to its additivity.

In particular, let measure μ be the Lebesgue measure λ .

Definition. Essential Lebesgue-measure support

$${}^{EL}S(A) = {}^{EL}\mathrm{supp}(A)$$

of a possibly non-measurable set A as such a subset of an abstract space S with the Lebesgue measure λ that has namely strictly positive outer measure

$$\lambda^*(A) > 0$$

is the set A without all its zero-measure-dense points so that

$$^{EL}S(A) = {}^{EL}supp(A) = A \setminus \{x \in A \mid d(x) = 0\}$$

with the same strictly positive outer measure

$$\lambda^*(^{EL}S(A)) = \lambda^*(A) > 0$$

due to its additivity.

The essence, sense, and meaning of this definition is as follows. Due to this set-theoretic difference, we exclude from set A all its zero-measure-dense points (not only isolated, but also possibly building parts locally zero-measure-dense and integrally zero-measure which have no influence on integration).

Nota bene: Excluding from set A all its namely directly zero-measure-dense points rather than zero-measure subsets is necessary because every point $x \in A$ is a zero-measure subset of set A so that

$$A \setminus \cup_{A' \subseteq A\, ,\, \lambda(A') = 0} A' = \varnothing \text{ (the empty set)}.$$

In practice, reducing namely Lebesgue measurable sets to their essential Lebesgue-measure supports is very simple but typical.

Example. Let us now provide reducing Lebesgue measurable set

$$A = [a\, , b] \cup [c\, , d] \cup Q \cup C$$

to their essential Lebesgue-measure support $^{EL}S(A) = {}^{EL}supp(A)$. Here

[a , b] and [c , d] are non-intersecting real-number segments with

$$a < b < c < d\ ;$$

Q is the set of all the rational numbers;

C is the Cantor ternary set [Smith, Cantor] built by consequently removing the open middle thirds of a line segment [0, 1], i.e.

$$C = [0, 1] \setminus (1/3, 2/3) \setminus (1/9, 2/9) \setminus (7/9, 8/9) \setminus \ldots$$

The Lebesgue measure extends the lengths of open, half-closed, and closed intervals [Lebesgue]. Therefore,

$$\lambda([a\, , b]) = b - a\, ,$$
$$\lambda([c\, , d]) = d - c\, .$$

Further,

$$\lambda(Q) = 0$$

because Q is countable only [Lebesgue, Halmos, Natanson, Kolmogorov & Fomin, Encyclopaedia of Mathematics].

$$\lambda(C) = 0,$$

too [Smith, Cantor], even if the Cantor ternary set has the continuum cardinality [Cantor].

Notata bene:

1. The set Q of all the rational numbers has a non-empty intersection with the union

$$[a\, , b] \cup [c\, , d]$$

of these mutually non-intersecting segments.

2. The Cantor ternary set C [Smith, Cantor] has a non-empty intersection with the union

$$[a\, , b] \cup [c\, , d]$$

of these mutually non-intersecting segments if and only if segment [0, 1] has a non-empty intersection with this union.

Therefore,

$$\{x \in A \mid d(x) = 0\} = Q \setminus ([a\, , b] \cup [c\, , d]) \cup C \setminus ([a\, , b] \cup [c\, , d]),$$
$$^{EL}S(A) = {}^{EL}supp(A) = A \setminus \{x \in A \mid d(x) = 0\} = [a\, , b] \cup [c\, , d],$$
$$\lambda(^{EL}S(A)) = \lambda(^{EL}supp(A)) = \lambda(A \setminus \{x \in A \mid d(x) = 0\})$$
$$= \lambda([a\, , b] \cup [c\, , d]) = \lambda([a\, , b]) + \lambda([c\, , d]) = b - a + d - c\, .$$

Notata bene:

1. Segments endpoints a , b , c , and d are NOT excluded because at each of them, the measure density is nonzero, namely 1/2.

2. If any of segments endpoints a , b , c , and d would be excluded already in the given set A , then the same would also hold for the essential Lebesgue-measure support $^{EL}S(A)$ whereas its measure

$\lambda(^{E}S(A))$ would remain the same.
3. If the namely open essential Lebesgue-measure support $(^{EL}S(A))$ is desired and/or required, then use

$$(^{EL}S(A)) = (a , b) \cup (c , d)$$

independently of including or excluding any of segments endpoints a , b , c , and d .
4. If the namely closed essential Lebesgue-measure support $(^{EL}S(A))'$ is desired and/or required, then use

$$(^{EL}S(A))' = [a , b] \cup [c , d]$$

independently of including or excluding any of segments endpoints a , b , c , and d .
5. Closing the given set A would be inadmissible even if the namely closed essential Lebesgue-measure support $(^{EL}S(A))'$ is desired and/or required. The reason is that closing the set Q of all the rational numbers gives the set R of all the real numbers

$$Q' = R .$$

Therefore, we would come to MISTAKE

$$^{EL}S(A) = (^{EL}S(A)) = (^{EL}S(A))' = R .$$

1.2. Essential Extrema and Bounds

In classical mathematics [Encyclopaedia of Mathematics], extrema (maximums and minimums) and bounds (suprema, or least upper bounds, as well as infima, or greatest lower bounds) are well-known. But they can take into account also integrally inessential, e.g. isolated, elements (points, values, etc.).
Before introducing the formal definitions of essential extrema and bounds versus common extrema and bounds, let us consider a typical example (Figure 1a).

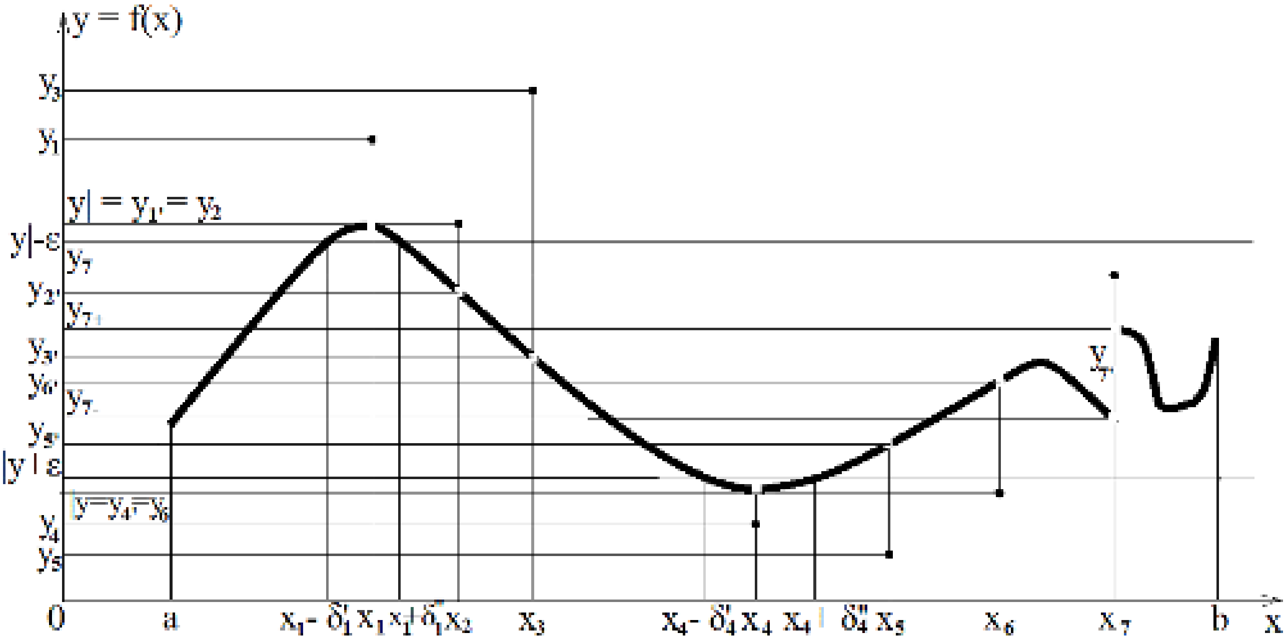

Figure 1a. Essential extrema and bounds versus common extrema and bounds

This one-argument real-domain real-range function

$$y = f(x)$$

with support [a , b] has the following removable and unremovable (inherent) discontinuities:
1) the removable discontinuity at

$$x = x_1 = x(1)$$

with the coinciding one-sided limits but another value at this point:

$$\lim_{x\to x(1)-} f(x) = y_{1'} = \lim_{x\to x(1)+} f(x)$$

but

$$f(x_1) = y_1 \neq y_{1'} ;$$

2) the removable discontinuity at

$$x = x_2 = x(2)$$

with the coinciding one-sided limits but another value at this point:

$$\lim_{x\to x(2)-} f(x) = y_{2'} = \lim_{x\to x(2)+} f(x)$$

but

$$f(x_2) = y_2 \neq y_{2'} ;$$

3) the removable discontinuity at

$$x = x_3 = x(3)$$

with the coinciding one-sided limits but another value at this point:

$$\lim_{x\to x(3)-} f(x) = y_{3'} = \lim_{x\to x(3)+} f(x)$$

but

$$f(x_3) = y_3 \neq y_{3'} ;$$

4) the removable discontinuity at

$$x = x_4 = x(4)$$

with the coinciding one-sided limits but another value at this point:

$$\lim_{x\to x(4)-} f(x) = y_{4'} = \lim_{x\to x(4)+} f(x)$$

but

$$f(x_4) = y_4 \neq y_{4'} ;$$

5) the removable discontinuity at

$$x = x_5 = x(5)$$

with the coinciding one-sided limits but another value at this point:

$$\lim_{x\to x(5)-} f(x) = y_{5'} = \lim_{x\to x(5)+} f(x)$$

but

$$f(x_5) = y_5 \neq y_{5'} ;$$

6) the removable discontinuity at

$$x = x_6 = x(6)$$

with the coinciding one-sided limits but another value at this point:

$$\lim_{x\to x(6)-} f(x) = y_{6'} = \lim_{x\to x(6)+} f(x)$$

but

$$f(x_6) = y_6 \neq y_{6'} ;$$

7) the unremovable (inherent) discontinuity at

$$x = x_7 = x(7)$$

with the coinciding one-sided limits but another value at this point:

$$\lim_{x\to x(7)-} f(x) = y_{7-} \neq y_{7+} = \lim_{x\to x(7)+} f(x)$$

and

$$y_{7-} \neq f(x_7) = y_7 \neq y_{7+} .$$

This function

$$y = f(x)$$

with support [a , b] has common maximum and least upper bound

$$\max_{[a\,,\,b]} f(x) = \sup_{[a\,,\,b]} f(x) = f(x_3) = y_3 ,$$

as well as common minimum and greatest lower bound

$$\min_{[a\,,\,b]} f(x) = \inf_{[a\,,\,b]} f(x) = f(x_5) = y_5 .$$

But they both are integrally inessential because do not coincide with any one-sided limits at these points x_3 and x_5 , respectively, and hence have no influence on the integral

$$\int_a^b f(x)dx$$

of this function f(x) on its support [a , b]. The reason is that they both are isolated points in the function graph set

$$\{(x , f(x)) \mid x \in [a , b]\}.$$

In no sufficiently small neighborhoods of these points x_3 and x_5 , there are any other points at which function f(x) takes values which are sufficiently near (close) to values y_3 and y_5 , respectively.

Namely, take any

$$\varepsilon \mid 0 < \varepsilon < y_3 - y_1 .$$

By no

$$\delta > 0,$$

the δ-neighborhood

$$x_3 - \delta < x < x_3 + \delta$$

contains any point

$$x \mid x \neq x_3 , f(x_3) - \varepsilon < f(x) < f(x_3) + \varepsilon .$$

The length and hence the measure of that δ-neighborhood are 2δ . The function graph set

$$\{(x , f(x)) \mid x \in [a , b]\}$$

has in open rectangle

$$[x_3 - \delta , x_3 + \delta] \times [f(x_3) - \varepsilon , f(x_3) + \varepsilon]$$

one point

$$(x_3 , f(x_3))$$

only. Its projection onto the x-axis is one point x_3 only. The common linear measure of any separate point vanishes. The same holds for the density

$$0/(2\delta) = 0$$

(in this δ-neighborhood) of the projections of all the near points onto the x-axis. Therefore,

$$y_3 = f(x_3)$$

is a zero-density common maximum and least upper bound.

Then take any

$$\varepsilon \mid 0 < \varepsilon < y_4 - y_5 .$$

By no

$$\delta > 0,$$

the δ-neighborhood

$$x_5 - \delta < x < x_5 + \delta$$

contains any point

$$x \mid x \neq x_5 , f(x_5) - \varepsilon < f(x) < f(x_5) + \varepsilon .$$

The length and hence the measure of that δ-neighborhood are 2δ . The function graph set

$$\{(x , f(x)) \mid x \in [a , b]\}$$

has in open rectangle

$$[x_5 - \delta , x_5 + \delta] \times [f(x_5) - \varepsilon , f(x_5) + \varepsilon]$$

one point

$$(x_5 , f(x_5))$$

only. Its projection onto the x-axis is one point x_3 only. The common linear measure of any separate point vanishes. The same holds for the density

$$0/(2\delta) = 0$$

(in this δ-neighborhood) of the projections of all the near points onto the x-axis. Therefore,

$$y_5 = f(x_5)$$

is a zero-density common minimum and greatest lower bound.

In our example,

$$y_{1'} = f(x_1) = \lim_{x\to x(1)-} f(x) = \lim_{x\to x(1)+} f(x)$$

meaningfully plays the role of the least upper bound, or the supremum. However, this is no maximum because function f(x) does not take this value $y_{1'}$ at this point x_1 (but occasionally takes this value $y_{1'}$ at another point x_2).

On the other hand,

$$y_{4'} = f(x_4) = \lim_{x\to x(4)-} f(x) = \lim_{x\to x(4)+} f(x)$$

meaningfully plays the role of the greatest lower bound, or the infimum. However, this is no minimum because function f(x) does not take this value $y_{4'}$ at this point x_4 (but occasionally takes this value $y_{4'}$ at another point x_6).

Let us now introduce the following formal definitions of essential extrema and bounds. To exclude their confusion with common extrema and bounds, let us use symbol ^{E} from the left.

Let μ be a measure and μ^* be an outer measure, e.g. the Lebesgue measure λ and outer measure λ^* , respectively.

Namely, let us denote and define the essential least upper bound, or the essential supremum, of one-argument real-domain real-range function

$$y = f(x)$$

with support [a , b] as

$$y| = {}^Ef|_{[a , b]} = {}^E\sup f(x)_{[a , b]} = \sup\{y \in R \mid \mu^*\{x \in [a , b] \mid f(x) \geq y\} > 0\}.$$

Further, let us denote and define the essential greatest lower bound, or the essential infimum, of

one-argument real-domain real-range function
$$y = f(x)$$
with support [a , b] as
$$|y = {}^{E}|f_{[a\,,\,b]} = {}^{E}\inf f(x)_{[a\,,\,b]} = \inf\{y \in R \mid \mu^*\{x \in [a\,,\,b] \mid f(x) \le y\} > 0\}.$$
Here
$$y| = {}^{E}f|_{[a\,,\,b]} = {}^{E}\sup f(x)_{[a\,,\,b]}$$
and
$$|y = {}^{E}|f_{[a\,,\,b]} = {}^{E}\inf f(x)_{[a\,,\,b]}$$
are designations of the essential least upper bound, or the essential supremum, as well as of the essential greatest lower bound, or the essential infimum, respectively;
sup and inf are the common least upper bound, or the common supremum, as well as the common greatest lower bound, or the common infimum, respectively [Encyclopaedia of Mathematics], here of the sets in the following braces, or curly brackets, {}.
Let us apply these both definitions, e.g., to our one-argument real-domain real-range function
$$y = f(x)$$
with support [a , b] (see Figure 1a) as follows.
To begin with, let us determine the essential least upper bound, or the essential supremum, of
$$y = f(x)$$
as
$$y| = {}^{E}f|_{[a\,,\,b]} = {}^{E}\sup f(x)_{[a\,,\,b]} = \sup\{y \in R \mid \mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} > 0\}.$$
If
$$y > y_3\,,$$
then
$$\{x \in [a\,,\,b] \mid f(x) \ge y\} = \varnothing \text{ (the empty set)},$$
$$\mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} = 0,$$
so condition
$$\mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} > 0$$
is not satisfied.
If
$$y_1 < y \le y_3\,,$$
then
$$\{x \in [a\,,\,b] \mid f(x) \ge y\} = \{x_3\},$$
$$\mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} = 0,$$
so condition
$$\mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} > 0$$
is not satisfied.
If
$$y| = y_{1'} = y_2 < y \le y_1\,,$$
then
$$\{x \in [a\,,\,b] \mid f(x) \ge y\} = \{x_1\,,\,x_3\},$$
$$\mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} = 0,$$
so condition
$$\mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} > 0$$
is not satisfied.
If
$$y = y| = y_{1'} = y_2\,,$$
then
$$\{x \in [a\,,\,b] \mid f(x) \ge y\} = \{x_1\,,\,x_2\,,\,x_3\},$$
$$\mu^*\{x \in [a\,,\,b] \mid f(x) \ge y\} = 0,$$
so condition

$$\mu^*\{x \in [a , b] \mid f(x) \geq y\} > 0$$

is not satisfied.

If

$$y = y| - \varepsilon = y_{1'} - \varepsilon = y_2 - \varepsilon$$

for any

$$0 < \varepsilon < y| - y_7 ,$$

then

$$\{x \in [a , b] \mid f(x) \geq y\} = [x_1 - \delta_1' , x_1 + \delta_1''] \cup \{x_1 , x_2 , x_3\}$$

where

$$\delta_1' > 0,$$
$$\delta_1'' > 0$$

and

$$\mu^*\{x \in [a , b] \mid f(x) \geq y\} = \delta_1' + \delta_1'' > 0,$$

so condition

$$\mu^*\{x \in [a , b] \mid f(x) \geq y\} > 0$$

is satisfied.

Therefore, the essential least upper bound, or the essential supremum, of our one-argument real-domain real-range function

$$y = f(x)$$

with support [a , b] is

$$y| = {}^{EL}f|_{[a , b]} = {}^{EL}\sup f(x)_{[a , b]}$$
$$= \sup\{y \in R \mid \mu^*\{x \in [a , b] \mid f(x) \geq y\} > 0\}$$
$$= \sup\{y = y| - \varepsilon \mid 0 < \varepsilon < y| - y_7\} = y| .$$

Let us now determine the essential greatest lower bound, or the essential infimum, of our one-argument real-domain real-range function

$$y = f(x)$$

with support [a , b] (see Figure 1a) as

$$|y = {}^{EL}|f_{[a , b]} = {}^{EL}\inf f(x)_{[a , b]} = \inf\{y \in R \mid \mu^*\{x \in [a , b] \mid f(x) \leq y\} > 0\}.$$

If

$$y < y_5 ,$$

then

$$\{x \in [a , b] \mid f(x) \leq y\} = \varnothing \text{ (the empty set)},$$
$$\mu^*\{x \in [a , b] \mid f(x) \leq y\} = 0,$$

so condition

$$\mu^*\{x \in [a , b] \mid f(x) \leq y\} > 0$$

is not satisfied.

If

$$y_5 \leq y < y_4 ,$$

then

$$\{x \in [a , b] \mid f(x) \leq y\} = \{x_5\},$$
$$\mu^*\{x \in [a , b] \mid f(x) \leq y\} = 0,$$

so condition

$$\mu^*\{x \in [a , b] \mid f(x) \leq y\} > 0$$

is not satisfied.

If

$$y_4 \leq y < |y = y_{4'} = y_6 ,$$

then

$$\{x \in [a , b] \mid f(x) \leq y\} = \{x_4 , x_5\},$$
$$\mu^*\{x \in [a , b] \mid f(x) \leq y\} = 0,$$

so condition

$$\mu^*\{x \in [a , b] \mid f(x) \le y\} > 0$$

is not satisfied.
If

$$y = |y = y_{4'} = y_6 ,$$

then

$$\{x \in [a , b] \mid f(x) \le y\} = \{x_4 , x_5 , x_6\},$$
$$\mu^*\{x \in [a , b] \mid f(x) \le y\} = 0,$$

so condition

$$\mu^*\{x \in [a , b] \mid f(x) \le y\} > 0$$

is not satisfied.
If

$$y = |y + \varepsilon = y_{4'} + \varepsilon = y_6 + \varepsilon ,$$

for any

$$0 < \varepsilon < y_{5'} - |y ,$$

then

$$\{x \in [a , b] \mid f(x) \le y\} = [x_4 - \delta_4' , x_4 + \delta_4''] \cup \{x_4 , x_5 , x_6\}$$

where

$$\delta_4' > 0,$$
$$\delta_4'' > 0$$

and

$$\mu^*\{x \in [a , b] \mid f(x) \le y\} = \delta_4' + \delta_4'' > 0,$$

so condition

$$\mu^*\{x \in [a , b] \mid f(x) \le y\} > 0$$

is satisfied.
Therefore, the essential greatest lower bound, or the essential infimum, of our one-argument real-domain real-range function

$$y = f(x)$$

with support [a , b] is

$$|y = {}^E f|_{[a , b]} = {}^E\inf f(x)_{[a , b]}$$
$$= \inf\{y \in R \mid \mu^*\{x \in [a , b] \mid f(x) \le y\} > 0\}$$
$$= \inf\{y = |y + \varepsilon \mid 0 < \varepsilon < y_{5'} - |y\} = |y .$$

Notata bene:
1. The common preliminary continuation of a function with discontinuities is useful if possible. For any one-argument real-domain real-range function with the both bounded one-sided limits at every point, replace the given function value with the half-sum (arithmetical mean) of the values of the both one-sided limits. At a point of continuity, the function remains continuous. At a point of removable discontinuity, the function changes and becomes continuous. At a point of unremovable (inherent) discontinuity, it remains. Such a continuation of our one-argument real-domain real-range function

$$y = f(x)$$

with support [a , b] (see Figure 1a) would give also one-argument real-domain real-range function

$$y = g(x) = [\lim_{t\to x-} f(t) + \lim_{t\to x+} f(t)]/2$$

with support [a , b]. These two functions differ at function f(x) discontinuity points only. Namely,

$$g(x_i) = f(x_{i'}) = y_{i'} = \lim_{x\to x(i)-} f(x) = \lim_{x\to x(i)+} f(x) = f(x_{i'}),\ i \in \{1, 2, ... , 6\},$$
$$g(x_7) = f(x_{7'}) = y_{7'} = [\lim_{x\to x(7)-} f(x) + \lim_{x\to x(7)+} f(x)]/2.$$

2. Our one-argument real-domain real-range function

$$y = f(x)$$

with support [a , b] (see Figure 1a) has namely removable discontinuities both at point x_1 providing the essential least upper bound, or the essential supremum, of f(x) and at point x_4 providing the essential greatest lower bound, or the essential infimum, of f(x). Therefore, the values of function

$$y = g(x)$$

continuating the given function

$$y = f(x)$$

at these both points provide the usual least upper bound, or the usual supremum, of continuated function $g(x)$ at point x_1 and the usual greatest lower bound, or the usual infimum, of continuated function $g(x)$ at point x_4 . Moreover, function $g(x)$ takes the corresponding values at these points, which provides at point x_1 namely the usual maximum and at point x_4 namely the usual minimum of continuated function $g(x)$.

3. Even if function $f(x)$ occasionally takes its essential least upper bound, or its essential supremum, $y|$ at point x_2 , this value and this point are NOT regarded as the essential maximum and a point of maximum, respectively.

4. Even if function $f(x)$ occasionally takes its essential greatest lower bound, or its essential infimum, $|y$ at point x_6 , this value and this point are NOT regarded as the essential minimum and a point of minimum, respectively.

5. At a point of unremovable (inherent) discontinuity, the greater one-sided limit may be the usual least upper bound, or the usual supremum (but no usual maximum), of continuated function $g(x)$ whereas the less one-sided limit may be the usual greatest lower bound, or the usual infimum (but no usual minimum), of continuated function $g(x)$. Then these one-sided limits provide both the essential least upper bound, or the essential supremum, and the essential greatest lower bound, or the essential infimum, respectively, of function $f(x)$. In such a case, both the given value of function $f(x)$ and the arithmetical mean value of continuated function $g(x)$ cannot play such roles, e.g. by our one-argument real-domain real-range function

$$y = f(x)$$

now with support $[x_6 , b]$ only rather than $[a , b]$ (see Figure 1a). Here:

usual maximum

$$y_{max} = y_7 = f(x_7)$$

is inessential;

usual minimum

$$y_{min} = y_6 = f(x_6)$$

is inessential;

the essential least upper bound, or the essential supremum, is

$$y| = {}^{E}f|_{[x(6),\, b]} = {}^{E}\sup f(x)_{[x(6),\, b]}$$
$$= \sup\{y \in R \mid \mu^*\{x \in [x_6 , b] \mid f(x) \geq y\} > 0\}$$
$$= f(x_7 + 0) = y_{7+} = \lim_{x\to x(7)+} f(x);$$

the essential greatest lower bound, or the essential infimum, is

$$|y = {}^{E}|f_{[x(6),\, b]} = {}^{E}\inf f(x)_{[x(6),\, b]}$$
$$= \inf\{y \in R \mid \mu^*\{x \in [x_6 , b] \mid f(x) \leq y\} > 0\}$$
$$= f(x_7 - 0) = y_{7-} = \lim_{x\to x(7)-} f(x).$$

1.3. Integration Development Necessity

In classical mathematics [encyclopaedia of mathematics], different integration methods [Riemann, Lebesgue, Saks, Natanson, Kolmogorov & Fomin, Shilov & Gurevich] are used, e.g.:
Riemann integration [Riemann];
Darboux integration [Darboux] equivalent to Riemann integration;
Lebesgue integration [Lebesgue];
Riemann-Stieltjes integration [Stieltjes] extending Riemann integration;
Lebesgue-Stieltjes integration [Lebesgue] extending Riemann-Stieltjes and Lebesgue integration and further developed by [Radon];
Daniell self-based integration scheme [Daniell] equivalent to Lebesgue integration and Lebesgue-Stieltjes integration;
narrow Denjoy integration [Denjoy 1912a, 1912b] and equivalent integration methods by [Perron], [Lusin], [Kurzweil], and [Henstock];
wide Denjoy integration [Denjoy 1916] and equivalent integration method by [Khintchine].
Namely integration methods by [Riemann], [Darboux], and [Lebesgue] play fundamental roles (possibly with [Stieltjes]).

1.4. Essential Integration

1.1. General Piecewise Function

Consider a general piecewise function. Using short (reduced) notation [Gelimson 2012a], piecewise represent any function $g_{Ra}(x_D)$ on domain D ($x \in D$) with range Ra as a domain of dependent variable g (value $g(x) \in Ra$). Use subfunctions $g_{Ra(j)}(x_{D(j)})$ on non-intersecting subdomains D_j ($x \in D_j$) with ranges Ra_j as follows:

$$g_{Ra}(x_D) = \cup_{j\in J}\, g_{Ra(j)}(x_{D(j)}).$$

Here

symbol $\cup$ unifies subfunctions on subdomains similarly to set theory and can be also indexed with an index set and range,

J is any (possibly uncountable) index set and range,

$$D(j) = D_j$$

are non-intersecting subdomains of domain D so that

$$D = \cup_{j\in J} D_j$$

whereas range Ra is the union of all its subranges Ra_j :

$$Ra = \cup_{j\in J} Ra_j .$$

Notata bene:

1. A partition, or non-intersecting distribution, of a domain between its subdomains is theoretically preferable.
2. However, it is possible that at a common point of continuity (e.g. at a subdomains boundary point) x of some subdomains

$$D_j = D(j) \ni x$$

with some subset

$$J_x \subseteq J$$

of indexes

$$j \in J_x ,$$

all the partial values $g_j(x)$ coincide and hence build common value g(x). Then it is admissible to explicitly include such point x into all these subdomains

$$D_j = D(j) \mid j \in J_x .$$

3. It is also possible that at a common point of continuity (e.g. at a subdomains boundary point) x of some subdomains

$$D_j = D(j) \ni x$$

with some subset

$$J_x \subseteq J$$

of indexes

$$j \in J_x ,$$

some partial values $g_j(x)$ for

$$j \in J_{x|g=0}$$

vanish whereas all the remaining partial values $g_j(x)$ for

$$j \in J_{x|g\neq 0}$$

coincide and hence build common nonzero value g(x). Then it is admissible to explicitly include such point x into all the subdomains

$$D_j = D(j) \mid j \in J_{x|g\neq 0} .$$

4. Support

$$S = \mathrm{supp}(g(x))$$

of a function g(x) is the set of all x (from domain D) for which g(x) is namely nonzero:

$$S = \mathrm{supp}(g(x)) = \{x \in D \mid g(x) \neq 0\},$$

$$S \subseteq D .$$

5. Extended support

$$S' \supseteq S$$

may be defined via weakening its requirement. Namely, require that

$$g(x) = 0 \mid x \in D \setminus S'$$

rather than

$$g(x) \neq 0 \mid x \in S'$$

so that g(x) may vanish at some points $x \in S'$.

6. Compact support [S] of a function g(x) is the smallest compact extension of support S , i.e. the intersection of all the compact extensions of support S .

7. Extended domain

$$D' \supseteq D$$

and extended range

$$Ra' \supseteq Ra$$

may be also used, especially if reducing extended domain D' to domain D and/or extended range Ra' to range Ra is obvious and unique, e.g. by excluding all the points of non-existing a function and/or by excluding all the values a function does not take, for instance by all the real numbers R :

$$y_{R'} = 1/x_{R'} ,$$

$$y_{R\setminus\{0\}} = 1/x_{R\setminus\{0\}} .$$

1.2. General Pointwise Function

Consider a pointwise function as a particular case of a piecewise function. Regard all the separate distinct elements of a domain (as a set) as its subdomains (subsets). Identify [Gelimson 2003a, 2003b] one-point set {x} at least here with this element (point) x itself. Use this element as an index whereas a whole domain as an index set. Then a pointwise function is as follows:

$$g_{Ra}(x_D) = \cup_{x\in D}\, g_{\{g(x)\}}(x_{\{x\}}),$$

or, simplifying,

$$g_{Ra}(x_D) = \cup_{x\in D}\, g_{g(x)}(x_x),$$

or simply

$$g(x_D) = \cup_{x\in D}\, g(x).$$

Here obvious domains {g(x)} = g(x) and {x} = x of variables g and x , respectively, can be omitted, D is any (possibly uncountable) index set, all one-element sets {x} are subdomains of domain D so that

$$D = \cup_{x\in D} \{x\}$$

whereas range Ra is the union of all its subranges Ra_j :

$$Ra = \cup_{x\in D} \{g(x)\}.$$

1.3. General Piecewise Probability Density

To consider namely a probability density function (distribution) f(x), take a non-negative real - valued function g(x), in our case a non-negative real-valued piecewise function

$$g_{Ra}(x_D) = \cup_{j\in J}\, g_{Ra(j)}(x_{D(j)}).$$

Here range Ra and all its subranges Ra_j are subsets of the set

$$R_0^+ = [0, +\infty)$$

of all the non-negative real numbers:

$$g_{Ra\subseteq[0, +\infty)}(x_D) = \cup_{j\in J}\, g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)}).$$

Further in order to provide a non-negative real-valued function $f_{Ra}(x_D)$ with the role of a probability density function, the integral normalization condition

$$\int_D f(x)\, dD = 1$$

has to be satisfied. Each of these both conditions (of non-negativity and normalization) are necessary, and their pair is sufficient for the possibility of f(x) to be a probability density function. Then, beginning with a non-negative real-valued piecewise function

$$g_{Ra\subseteq[0, +\infty)}(x_D) = \cup_{j\in J}\, g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)})$$

with namely positive integral

$$\int_D g(x)\, dD > 0,$$

simply divide this function g(x) by this integral to obtain a probability density function:

$$f_{Ra\subseteq[0, +\infty)}(x_D) = g_{Ra\subseteq[0, +\infty)}(x_D)/\int_D g_{Ra\subseteq[0, +\infty)}(x_D)\, dD = \cup_{j\in J}\, g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)})/\Sigma_{j\in J} \int_{D(j)} g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)})\, dD_j\ .$$

Support S of a non-negative real-valued function g(x) is the set of all x (from domain D) for which g(x) is namely strictly positive:

$$S = \{x \in D \mid g(x) > 0\},$$

$$S \subseteq D\ .$$

Compact support [S] of a non-negative real-valued function g(x) is the smallest compact extension of support S , i.e. the intersection of all the compact extensions of support S .

By integration, we may simply replace domain D and subdomains D_j :

1) either via compact support [S] and compact subsupports $[S_j]$, respectively,

2) or via support S and subsupports S_j , respectively.

Then we have, e.g.,

$$\int_S f(x)\, dS = 1,$$

$$\int_{[S]} f(x)\, d[S] = 1,$$

$$\int_S g(x)\, dS > 0,$$

$$\int_{[S]} g(x)\, d[S] > 0,$$

$$f_{Ra\subseteq[0, +\infty)}(x_D) = g_{Ra\subseteq[0, +\infty)}(x_D)/\int_S g_{Ra\subseteq[0, +\infty)}(x_D)\, dS = \cup_{j\in J}\, g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)})/\Sigma_{j\in J} \int_{S(j)} g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)})\, dS_j\ ,$$

$$f_{Ra\subseteq[0, +\infty)}(x_D) = g_{Ra\subseteq[0, +\infty)}(x_D)/\int_{[S]} g_{Ra\subseteq[0, +\infty)}(x_D)\, d[S] = \cup_{j\in J}\, g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)})/\Sigma_{j\in J} \int_{[S(j)]} g_{Ra(j)\subseteq[0, +\infty)}(x_{D(j)})\, d[S_j].$$

1.4. General Pointwise Probability Density

To consider namely a probability density function (distribution) f(x), take a non-negative real-valued function, in our case a non-negative real-valued pointwise function

$$g_{Ra\subseteq[0, +\infty)}(x_D) = \cup_{x\in D}\, g_{\{g(x)\}\subseteq[0, +\infty)}(x_{\{x\}}),$$

or, simplifying,

$$g_{Ra\subseteq[0, +\infty)}(x_D) = \cup_{x\in D}\, g_{g(x)}(x_x),$$

or simply

$$g(x_D) = \cup_{x\in D}\, g(x).$$

Consider a pointwise function as a particular case of a piecewise function. Regard all the separate distinct elements of a domain (as a set) as its subdomains (subsets). Identify [Gelimson 2003a, 2003b] one-point set {x} at least here with this element (point) x itself. Use this element as an index whereas a whole domain as an index set.

Here obvious domains {g(x)} = g(x) and {x} = x of variables g and x , respectively, can be omitted, D is any (possibly uncountable) index set, all one-element sets {x} are subdomains of domain D so that

$$D = \cup_{x\in D}\, \{x\}$$

whereas range Ra is the union of all its subranges {g(x)}:

$$Ra = \cup_{x\in D}\, \{g(x)\}.$$

Range Ra and all its subranges {g(x)}are subsets of the set

$$R_0^+ = [0, +\infty)$$

of all the non-negative real numbers:

$$g_{Ra\subseteq[0, +\infty)}(x_D) = \cup_{j\in J}\, g_{\{g(x)\}\subseteq[0, +\infty)}(x_{\{x\}}).$$

Further in order to provide a non-negative real-valued function $f_{Ra}(x_D)$ with the role of a probability density function, the integral normalization condition

$$\int_D f(x)\, dD = 1$$

has to be satisfied. Each of these both conditions (of non-negativity and normalization) are necessary, and their pair is sufficient for the possibility of f(x) to be a probability density function. Then, beginning with a non-negative real-valued piecewise function

$$g_{Ra\subseteq[0, +\infty)}(x_D) = \cup_{x\in D}\, g_{\{g(x)\}\subseteq[0, +\infty)}(x_{\{x\}})$$

with namely positive integral

$$\int_D g(x)\, dD > 0,$$

simply divide this function g(x) by this integral to obtain a probability density function:

$$f_{Ra\subseteq[0, +\infty)}(x_D) = g_{Ra\subseteq[0, +\infty)}(x_D)/\int_D g_{Ra\subseteq[0, +\infty)}(x_D)\, dD$$
$$= \cup_{x\in D}\, g_{\{g(x)\}\subseteq[0, +\infty)}(x_{\{x\}})/\int_D g_{Ra\subseteq[0, +\infty)}(x_D)\, dD = \cup_{x\in D}\, g(x)/\int_D g(x)\, dD\, .$$

Nota bene: Here unions may be also uncountable.

By integration, we may simply replace domain D and subdomains D_j :

1) either via compact support [S] and compact subsupports $[S_j]$, respectively,

2) or via support S and subsupports S_j , respectively.

Then we have, e.g.,

$$\int_S f(x)\, dS = 1,$$
$$\int_{[S]} f(x)\, d[S] = 1,$$
$$\int_S g(x)\, dS > 0,$$
$$\int_{[S]} g(x)\, d[S] > 0,$$
$$f_{Ra\subseteq[0, +\infty)}(x_D) = g_{Ra\subseteq[0, +\infty)}(x_D)/\int_S g_{Ra\subseteq[0, +\infty)}(x_D)\, dS$$
$$= \cup_{x\in D}\, g_{\{g(x)\}\subseteq[0, +\infty)}(x_{\{x\}})/\int_S g_{Ra\subseteq[0, +\infty)}(x_D)\, dS = \cup_{x\in D}\, g(x)/\int_S g(x)\, dS\, ,$$
$$f_{Ra\subseteq[0, +\infty)}(x_D) = g_{Ra\subseteq[0, +\infty)}(x_D)/\int_{[S]} g_{Ra\subseteq[0, +\infty)}(x_D)\, d[S]$$
$$= \cup_{x\in D}\, g_{\{g(x)\}\subseteq[0, +\infty)}(x_{\{x\}})/\int_{[S]} g_{Ra\subseteq[0, +\infty)}(x_D)\, d[S] = \cup_{x\in D}\, g(x)/\int_{[S]} g(x)\, d[S].$$

1.5. Particular Case Hierarchy

Domain D , support S , compact support [S], their partitions into subdomains D_j , subsupports S_j , compact subsupports $[S_j]$, respectively, as well as a non-negative real-valued piecewise function g(x) with namely positive integral on D , may be arbitrary. Therefore, we have here a multidimensional hierarchy of particular cases.

1. Domain D may be, e.g., discrete, continual, or mixed (with both discrete and continual parts). These simplest possibilities are especially typical in practice. Domain D may be, in particular, a subset of a countable-dimensional Euclidean space or of finite-dimensional Euclidean space R^n ($n \in N = \{1, 2, ...\}$), e.g. of one-dimensional Euclidean space $R = (-\infty , \infty)$. Naturally, continual domain D may be, in particular, one of these spaces as a whole.
2. Support S may be any subset of domain D .
3. Compact support [S] is the smallest compact extension of support S , i.e. the intersection of all the compact extensions of support S . The Bolzano-Weierstrass theorem [Encyclopaedia of Mathematics] proved that in any Euclidean space, a set is compact if and only if it is both closed and bounded. Then a compact support coincides with the corresponding closed support, or the support closure.
4. Partition of domain D into subdomains D_j may be arbitrary: from using domain D itself with no partition to separating every point $x \in D$. In any Euclidean space, partitioning domain D by every coordinate into a finite set of non-intersecting intervals (including open intervals, half-open intervals, and segments as closed intervals) at least partially containing their endpoints is typical if possible. These simplest of the additive Borel sets provide the identity of all the common measures [Cramér, Encyclopaedia of Mathematics] and are preferable in probability theory, too.
5. Partition of support S into subsupports S_j may be arbitrary: from using support S itself with no partition to separating every point $x \in S$. In any Euclidean space, partitioning namely bounded support S by every coordinate into a finite set of non-intersecting intervals (including open intervals, half-open intervals, and segments as closed intervals) at least partially containing their endpoints is typical if possible. These simplest of the additive Borel sets provide the identity of all the common measures [Cramér, Encyclopaedia of Mathematics] and are preferable in probability theory, too.
6. Partition of compact support [S] into subsupports $[S_j]$ may be arbitrary: from using support [S] itself with no partition to separating every point $x \in [S]$. In any Euclidean space, partitioning namely bounded compact support [S] by every coordinate into a finite set of segments containing their endpoints and possibly intersecting at them is typical if possible. These simplest of the additive Borel sets provide the identity of all the common measures [Cramér, Encyclopaedia of Mathematics] and are preferable in probability theory, too.
7. A non-negative real-valued piecewise function g(x) with namely positive integral on D may be arbitrary. To provide integrability of probability density function f(x) also multiplied by desired and/or required powers of variable x to obtain explicit (closed-form) integral (cumulative) probability distribution function F(x) along with moments [Cramér, Encyclopaedia of Mathematics], use namely the simplest and most suitable classes of functions to piecewise build a desired and/or required non-negative real-valued function g(x). Among them are, e.g., some power functions including polynomials, rational, exponential, trigonometric, and hyperbolic functions, as well as their linear and nonlinear combinations. Such function variety and partition variety provide very many possibilities of solving typical classes of urgent problems also in probability theory and mathematical statistics.

Therefore, namely the simplest piecewise linear functions whose domain D is a one-dimensional space $R = (-\infty , \infty)$ and whose support S is bounded and representable via a finite set of non-intersecting intervals (including open intervals, half-open intervals, and segments as closed intervals) at least partially containing their endpoints are further especially elaborately considered in the present work. They provide adequately fitting practically any desired and/or required function with any desired and/or required precision.

1.6. Non-Strictly Monotonic Function Continualization and Inversion

Inverting a strictly monotonic function (which is a bijection, or bijective function, or one-to-one correspondence) is well-known [Encyclopaedia of Mathematics] and straightforward because every image (output) always has namely the only preimage (input). But the common notation of the inverse is not suitable in general. For example, the inverse to function

$$y = h(x)$$

is usually denoted by

$$x = h^{-1}(y)$$

with always necessary explanation that $h^{-1}(y)$ means here NOT

$$1/h(y),$$

which would be natural, but the function which is inverse to function h . Using exponent -1 in such a manner makes sense in the unique case, namely for the multiplicative inverse, or reciprocal, a^{-1} to element a , e.g. by numbers or matrices. In practically every context with inversion, this operation can be confused with exponentiation. By inversion in general, using the common reciprocal form makes not much sense. Further using the exponent gives a formula an additional higher level by notation. Therefore, expressions with inverses in indexes and exponents cannot be properly shown in many cases and hence are not suitable at all. However, this common notation of the inverse is traditional and could be further used along with a more suitable notation.
[Gelimson 2012b] proposed a possible alternative notation of the inverse. For example, the inverse to function

$$y = h(x)$$

is denoted by

$$x = \underline{h}(y)$$

using the underline with no necessity of revocation and without creating an additional level. Simultaneously adding the traditional notation via

$$x = \underline{h}(y) = h^{-1}(y)$$

has some obvious advantages, too. It gives an aid to remember the sense of this new notation via building an association with the common notation which helps here this new notation. On the other hand, the new notation helps here the common notation. The reason is that using $\underline{h}(y)$ brings here at least doubt in the generally false reciprocal role of inversion.
Let the same one-argument one-value real-number function

$$y = h(x)$$

be namely non-strictly monotonic. Then there are at least two different values x_1 and x_2 of argument x such that

$$x_1 < x_2$$

but

$$y_1 = h(x_1) = h(x_2) = y_2 .$$

Let us denote their common value via

$$y_{12} = y_1 = y_2 .$$

Then image y_{12} has at least two distinct preimages x_1 and x_2 . Determine and gather all the distinct preimages of the same image y_{12} . They build a set which may be denoted by

$$\underline{h}(y_{12}).$$

In particular, this set contains both x_1 and x_2 . Further there exist both the greatest lower bound

$$\underline{h}_{inf}(y_{12})$$

and the least upper bound

$$\underline{h}_{sup}(y_{12})$$

of this set. It seems to be better to denote them via $|\underline{h}$ and $\underline{h}|$, respectively. Namely,

$$|\underline{h}(y_{12}) = \underline{h}_{inf}(y_{12}),$$

$$\underline{h}|(y_{12}) = \underline{h}_{sup}(y_{12}).$$

If the both bounds are truly taken, then they are the minimal and the maximal elements of this set, respectively, with natural notation

$$\underline{h}_{min}(y_{12})$$

and

$$\underline{h}_{max}(y_{12}).$$

Then

$$|\underline{h}(y_{12}) = \underline{h}_{min}(y_{12}) = \underline{h}_{inf}(y_{12}),$$
$$\underline{h}|(y_{12}) = \underline{h}_{max}(y_{12}) = \underline{h}_{sup}(y_{12}).$$

Analyze in such a manner every image

$$y = y_{12}$$

with at least two distinct preimages x_1 and x_2 .
Further for every image y with the only preimage x ,

$$|\underline{h}(y) = \underline{h}_{min}(y) = \underline{h}_{inf}(y) = x$$

and

$$\underline{h}|(y) = \underline{h}_{max}(y) = \underline{h}_{sup}(y) = x .$$

Therefore, there exist the infimum (or greatest lower bound) inverse function

$$x = |\underline{h}(y)$$

and the supremum (or greatest lower bound) inverse function

$$x = \underline{h}|(y).$$

These one-argument one-value real-number functions are namely the both extreme functions among all the generally many-valued functions inverse to generally non-strictly monotonically increasing one-argument one-value real-number function

$$y = h(x).$$

Then there is an open interval, one of two half-open and half-closed intervals, or a segment as a closed interval

$$(|x , x|), (|x , x|], [|x , x|), [|x , x|],$$

or

$$(|\underline{h}(y) , \underline{h}|(y)), (|\underline{h}(y) , \underline{h}|(y)], [|\underline{h}(y) , \underline{h}|(y)), [|\underline{h}(y) , \underline{h}|(y)],$$

whose either excluded or included (independently from one another) endpoints are

$$|x = |\underline{h}(y)$$

and

$$x| = \underline{h}|(y)$$

and which may be regarded as the total preimage of image y .

1.7. Integral (Cumulative) Probability Distribution Function Inversion

Let domain D of integral (cumulative) probability distribution function F(x) with range Ra = [0, 1] be one-dimensional Euclidean space R = $(-\infty , \infty)$. Then F(x) is a one-argument one-value real-number function. It can be not only strictly monotonically increasing, but also locally non-strictly monotonically increasing. If its arbitrary image y belonging to range Ra = [0, 1] has at least two distinct preimages x_1 and x_2 , then there is an open interval, one of two half-open and half-closed intervals, or a segment as a closed interval

$$(|x , x|), (|x , x|], [|x , x|), [|x , x|],$$

or

$$(|\underline{F}(y), \underline{F}|(y)), (|\underline{F}(y), \underline{F}|(y)], [|\underline{F}(y), \underline{F}|(y)), [|\underline{F}(y), \underline{F}|(y)],$$

whose either excluded or included (independently from one another) endpoints are

$$|x = |\underline{F}(y)$$

and

$$x| = \underline{F}|(y)$$

and which may be regarded as the total preimage of image y .

Nota bene: On this interval possibly excluding its subset of zero measure, probability density function f(x) vanishes. Otherwise, integral (cumulative) probability distribution function y = F(x) could not be constant on this interval.

Therefore, there exist the infimum inverse function

$$x = |\underline{F}(y)$$

and the supremum inverse function

$$x = \underline{F}|(y).$$

These one-argument one-value real-number functions are namely the both extreme functions among all the generally many-valued functions inverse to generally non-strictly monotonically increasing one-argument one-value real-number function

$$y = F(x).$$

1.8. Generally Non-Strictly Monotonic Sequence Continualization and Inversion

Let domain D of function g(x) be one-dimensional Euclidean space:

$$D = R = (-\infty , \infty).$$

Let namely finite real-number segment

$$S' = [a , b] \subset R = (-\infty , \infty) \ (-\infty < a < b < \infty)$$

be an extended support of function g(x) so that

$$g(x) = 0$$

at any

$$x \in R \setminus S' = (-\infty , a) \cup (b , \infty).$$

Let further n ($n \in N = \{1, 2, ...\}$) intermediate points $c_1 , c_2 , c_3 , c_4 , \dots , c_{n-3} , c_{n-2} , c_{n-1} , c_n$ in the non-decreasing order (to provide passages to limits with changing the number n of these points) so that

$$a \le c_1 \le c_2 \le c_3 \le c_4 \le \dots \le c_{n-3} \le c_{n-2} \le c_{n-1} \le c_n \le b$$

divide this segment into n + 1 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a ,$$

$$c_{n+1} = b ,$$

$$c(i) = c_i \ (i = 0, 1, 2, \dots , n + 1).$$

Let us further generalize the last notation, namely for a generally non-strictly monotonic sequence as function c(z) of index z

$$c(z) = c_z \ (z = z' , z' + 1, z' + 2, z' + 3, z' + 4, z' + 5, \dots , z'' , z'' + 1, \dots , z''')$$

(Fig. 1).

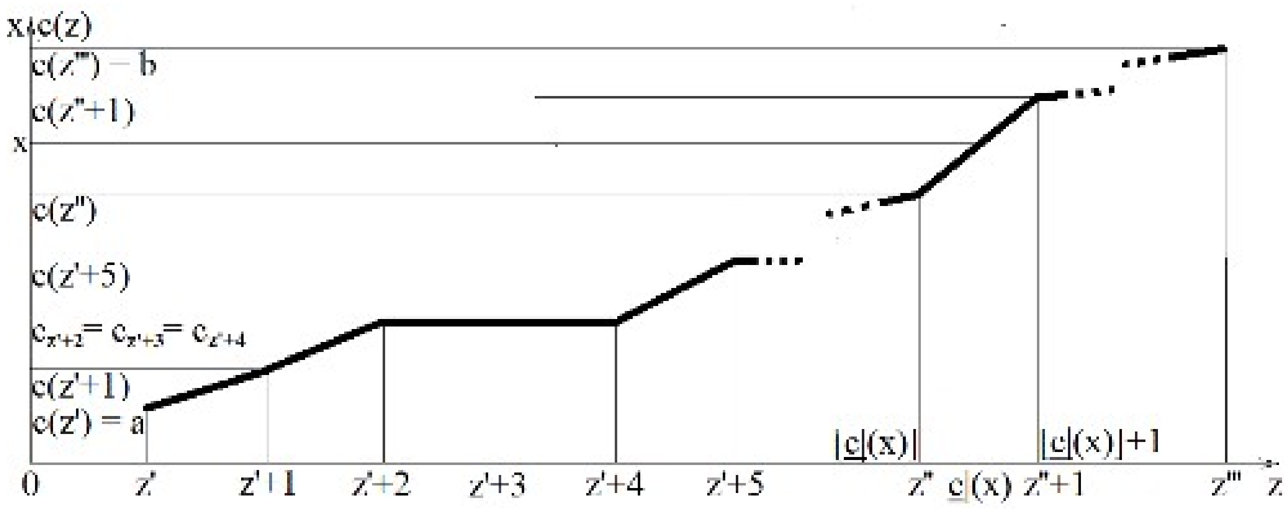

Figure 1. Generally non-strictly monotonic sequence continualization and inversion

The numeration begins at any initial integer

$$z' \in Z = \{... , -2, -1, 0, 1, 2, ...\}$$

and ends at any final greater integer

$$z''' \in Z = \{... , -2, -1, 0, 1, 2, \dots\},$$

$$z' < z''' .$$

In the above particular case

$$z' = 0,$$

$$z''' = n + 1.$$

In our general case

$$a = c_{z'} \le c_{z'+1} \le c_{z'+2} \le c_{z'+3} \le c_{z'+4} \le c_{z'+5} \le \ldots \le c_{z''} \le c_{z''+1} \le \ldots \le c_{z'''} = b\ .$$

Nota bene: By namely strictly monotonically increasing sequence

$$c(z) = c_z\ ,$$

all these inequalities would be strict:

$$a = c_{z'} < c_{z'+1} < c_{z'+2} < c_{z'+3} < c_{z'+4} < c_{z'+5} < \ldots < c_{z''} < c_{z''+1} < \ldots < c_{z'''} = b\ .$$

In this case only, all the z''' - z' + 1 points

$$a = c_{z'}\ , c_{z'+1}\ , c_{z'+2}\ , c_{z'+3}\ , c_{z'+4}\ , c_{z'+5}\ , \ldots , c_{z''}\ , c_{z''+1}\ , \ldots , c_{z'''} = b$$

would be namely distinct. However, in our general case, some of these z''' - z' + 1 points may coincide. In Figure 1, e.g., at least

$$c_{z'+2} = c_{z'+3} = c_{z'+4}$$

and

$$c_{z'+5} = c_{z'+6}$$

even if index z' + 6 is not explicitly shown but the direction from shown point $c_{z'+5}$ to omitted point $c_{z'+6}$ is obviously horizontal.

We shall use inverting this generally non-strictly monotonic sequence as function c(z) of index z also to provide short (reduced) notation of integral (cumulative) probability distribution function F(x) with domain

$$R = (-\infty\ , \infty)$$

(one-dimensional Euclidean space) and range

$$Ra = [0, 1].$$

Then F(x) is a one-argument one-value real-number function. It can be not only strictly monotonically increasing, but also locally non-strictly monotonically increasing. Further there are similar problems with inverting integral (cumulative) probability distribution function F(x), in particular with determining quantiles, e.g. medians, quartiles, deciles, and percentiles. All this is our general aim. And our particular task here is as follows. For any real

$$x \in S' = [a\ , b] \subset R = (-\infty\ , \infty)\ (-\infty < a < b < \infty),$$

explicitly (closed-form) determine index z'' of left-closed and right-open interval

$$[c_{z''}\ , c_{z''+1}) = [c(z''), c(z'' + 1))$$

containing this x so that

$$x \in [c_{z''}\ , c_{z''+1}) = [c(z''), c(z'' + 1)),$$
$$c_{z''} \le x < c_{z''+1}\ ,$$
$$c(z'') \le x < c(z'' + 1).$$

Notata bene:

1. If the set of all the points c(z) for which

$$c(z) = x$$

contains the only point, then denote this index z via z'' :

$$z'' = z \mid z \in \{z'\ , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \ldots , z'''\}, c(z) = x\ .$$

As applied to such point x , this generally non-strictly monotonic sequence as function c(z) of index z behaves so as if this sequence were strictly monotonic.

2. If the set of all the points c(z) for which

$$c(z) = x$$

contains more than one point, then select namely the point with the maximal index z among the indexes of all these points c(z) and denote this maximal index z via z'' :

$$z'' = \max\{z \in \{z'\ , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \ldots , z'''\} \mid c(z) = x\}.$$

3. If for real

$$x \in S' = [a\ , b] \subset R = (-\infty\ , \infty)\ (-\infty < a < b < \infty),$$

there is no point c(z) for which

$$c(z) = x\ ,$$

then it is also possible to explicitly (closed-form) determine index z'' of left-closed and right-open interval

$$[c_{z''} , c_{z''+1}) = [c(z''), c(z''+1))$$

containing this x so that

$$x \in [c_{z''} , c_{z''+1}) = [c(z''), c(z''+1)),$$
$$c_{z''} \le x < c_{z''+1} ,$$
$$c(z'') \le x < c(z''+1).$$

Let us piecewise linearly continualize sequence c(z) via simply consequently connecting all the z''' - z' + 1 diagram points in Figure 1

$$(z' , c_{z'}), (z' + 1, c_{z'+1}), (z' + 2, c_{z'+2}), (z' + 3, c_{z'+3}), (z' + 4, c_{z'+4}), (z' + 5, c_{z'+5}), \ldots , (z''' c_{z'''}).$$

Namely, on every real-number segment

$$[z , z + 1]$$

for any integer

$$z \in \{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \ldots , z''' - 1\},$$

apply linear interpolation

$$d(t) = c(z)(z + 1 - t) + c(z + 1)(t - z)$$
$$= c_z(z + 1 - t) + c_{z+1}(t - z).$$

Notata bene:

1. For sequence c(z) which is an integer-argument real-valued function, its discrete domain is

$$\{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \ldots , z'''\}$$

containing these z''' - z' + 1 integer points only.

2. For real-argument real-valued function d(t), its continual domain is real segment

$$[z' , z''']$$

also containing all the intermediate (internal) real points.

3. At every

$$z \in \{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \ldots , z'''\}$$

of these z''' - z' + 1 integer points, real-argument real-valued function d(t) coincides with sequence c(z) which is an integer-argument real-valued function.

Therefore, real-argument real-valued function d(t) is a generalization of sequence c(z) which is an integer-argument real-valued function with extending it from discrete domain

$$\{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \ldots , z'''\}$$

to continual domain

$$[z' , z''']$$

also containing all the intermediate (internal) real points.

4. We may piecewise continualize sequence c(z) not only linearly but also non-linearly with always piecewise conserving the monotonicity properties of sequence c(z) on every real-number segment

$$[z , z + 1]$$

for any integer

$$z \in \{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \ldots , z''' - 1\}.$$

Namely, real-argument real-valued continuous function d(t) is:

4.1) strictly increasing on real-number segment [z , z + 1] with taking not only the values

$$d(z) = c_z$$

and

$$d(z + 1) = c_{z+1}$$

at the endpoints z and z + 1, but also all the intermediate real values between c_z and c_{z+1} at the corresponding points between z and z + 1, if and only if

$$c_z < c_{z+1} ;$$

4.2) constant

$$d(t) = c_z = c_{z+1}$$

on real-number segment [z , z + 1] if and only if

$$c_z = c_{z+1} .$$

5. Therefore, if and only if

$$c_z = c_{z+1} ,$$

so that these points namely coincide, then the only possibility is to apply linear interpolation

$$d(t) = c_z(z + 1 - t) + c_{z+1}(t - z) =$$
$$= c_z(z + 1 - t) + c_z(t - z) = c_z = c_{z+1}$$

only (here even constant).

6. Otherwise, i.e. if and only if

$$c_z < c_{z+1} ,$$

so that these points are namely distinct, then along with linear interpolation

$$d(t) = c(z)(z + 1 - t) + c(z + 1)(t - z)$$
$$= c_z(z + 1 - t) + c_{z+1}(t - z),$$

there are infinitely many different possibilities to apply nonlinear interpolation.

7. Moreover, due to nonlinear interpolation, there are infinitely many different possibilities to provide not only the continuity (which is the case by linear interpolation) of continualized function d(t), but also its differentiability.

8. To provide the differentiability of continualized function d(t) at such integer point

$$z \in \{z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \dots , z''' - 1\}$$

that

$$c_{z-1} = c_z < c_{z+1} ,$$

it is necessary that the derivative of continualized function d(t) at this integer point from the right vanishes:

$$d'(z + 0) = d'(z - 0) = 0.$$

9. To provide the differentiability of continualized function d(t) at such integer point

$$z \in \{z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \dots , z''' - 1\}$$

that

$$c_{z-1} < c_z = c_{z+1} ,$$

it is necessary that the derivative of continualized function d(t) at this integer point from the left vanishes:

$$d'(z - 0) = d'(z + 0) = 0.$$

10. To provide the differentiability of continualized function d(t) at such integer point

$$z \in \{z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \dots , z''' - 1\}$$

that

$$c_{z-1} < c_z < c_{z+1} ,$$

i.e. z is a point of the two-sided strict increase of continualized function d(t), it is necessary that the derivatives of continualized function d(t) at this integer point from the left and from the right both have a common non-negative value:

$$d'(z - 0) = d'(z + 0) \geq 0.$$

11. To provide the differentiability of continualized function d(t) at every integer point

$$z \in \{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \dots , z'''\},$$

it is sufficient that the derivative of continualized function d(t) at this integer point both from the left and from the right vanishes:

$$d'(z) = d'(z - 0) = d'(z + 0) = 0.$$

12. If

$$c_z < c_{z+1} ,$$

then there are infinitely many different possibilities to apply such nonlinear interpolation, e.g.

$$d(t) = (c_{z+1} - c_z)/2\ |\sin[\pi(t - z - 1/2)]|^u \operatorname{sign}(t - z - 1/2) + (c_z + c_{z+1})/2$$

for real segment [z , z + 1] by any integer

$$z \in \{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \dots , z''' - 1\}$$

and for any real

$$u \geq 1.$$

13. This curve

$$\{(t , d(t)) \mid t \in [z , z + 1]\}$$

is central symmetric about its inflection point

$$(z + 1/2, (c_z + c_{z+1})/2),$$

which is generally not necessary.

14. The inflection point of such a curve may be any in open rectangular

$$(z , z + 1) \times (c_z , c_{z+1})$$

without any symmetricity.

15. If

$$c_z < c_{z+1} ,$$

then there are infinitely many different possibilities to apply such nonlinear interpolation, e.g. via arbitrarily dividing each of the both pieces

$$[z , z + 1]$$

and

$$[c_z , c_{z+1}]$$

into two subpieces

$$[z , z + q], [z + q , z + 1]$$

and

$$[c_z , c_z + r(c_{z+1} - c_z)], [c_z + r(c_{z+1} - c_z), c_{z+1}],$$

respectively. This is equivalent to selecting any real

$$q \mid 0 < q < 1$$

and

$$r \mid 0 < r < 1.$$

Further define

$$d(t) = c_z + r(c_{z+1} - c_z)[(t - z)/q]^u , t \in [z , z + q],$$

$$d(t) = c_{z+1} - (1 - r)(c_{z+1} - c_z)[(z + 1 - t)/(1 - q)]^v , t \in [z + q , z + 1]$$

for real segment $[z , z + 1]$ by any integer

$$z \in \{z' , z' + 1, z' + 2, z' + 3, z' + 5, z' + 5, \dots , z''' - 1\}.$$

and for any real

$$u \geq 1.$$

Hence we may rename function d to c and argument t to z and simply deal with real-argument real-valued function c(z).

Now we can solve our problem when for real

$$x \in S' = [a , b] \subset R = (-\infty , \infty) (-\infty < a < b < \infty),$$

there is no point c(z) for which

$$c(z) = x .$$

Apply inverting real-argument real-valued function c(z) to explicitly (closed-form) determine index z'' of left-closed and right-open interval

$$[c_{z''} , c_{z''+1}) = [c(z''), c(z''+1))$$

containing this x so that

$$x \in [c_{z''} , c_{z''+1}) = [c(z''), c(z''+1)),$$

$$c_{z''} \leq x < c_{z''+1} ,$$

$$c(z'') \leq x < c(z''+1).$$

Denote the inverse function to real-argument real-valued function

$$x = c(z)$$

via

$$z = \underline{c}(x).$$

1. If the set of all the points c(z) for which

$$c(z) = x$$

contains the only point, then inverse function

$$z = \underline{c}(x)$$

provides the desired and required abscissa z .

2. If the set of all the points c(z) for which

$$c(z) = x$$

contains more than one point, then select namely the point with the maximal index z among the indexes of all these points c(z). To provide this, take namely the supremum inverse function

$$z = \underline{c}|(x).$$

However, we need namely index z" of left-closed and right-open interval

$$[c_{z''} , c_{z''+1}) = [c(z''), c(z''+1))$$

containing this x so that

$$x \in [c_{z''} , c_{z''+1}) = [c(z''), c(z''+1)),$$

$$c_{z''} \leq x < c_{z''+1} ,$$

$$c(z'') \leq x < c(z''+1).$$

Using namely the supremum inverse function

$$z = \underline{c}|(x),$$

simply determine

$$z'' = \lfloor \underline{c}|(x) \rfloor.$$

Here the floor (or entier) function [Encyclopaedia of Mathematics]

$$v = \lfloor w \rfloor$$

of real argument w gives the largest integer v less than or equal to w .

2. Piecewise Probability Density

2.1. Main Definitions

Consider a general one-dimensional bounded-support finite-piecewise probability density (Fig. 2). Additionally suppose that it and its products with natural powers (to ensure existing the following initial and hence also central moments of the desired and/or required orders) of the independent variable are integrable. To provide this, probability density continuity on each peace (here interval) is sufficient. If a general one-dimensional bounded-support probability density has on its support a finite number of discontinuity points, which typically holds in practice, take namely all these points along with the support endpoints to partition the support into the peaces (here intervals). If the analytical expressions of the probability density change at some internal (i.e. continuity) points on the intervals, then additionally include these points into the set of the partitioning points. By adding such probability densities together, simply unify their sets of the partitioning points. If some of such probability densities have intersecting supports, pointwise add such probability densities together. Then normalize (as follows) the obtained probability quasidensity to ensure namely a probability density with the unit integral.
Nota bene: For a probability itself [Loève] (rather than a probability density as a very suitable mean or instrument for probability investigation) especially important in practice, use namely Lebesgue-Stieltjes-integral and Lebesgue-integral [Lebesgue, Stieltjes] (rather than Riemann-integral [Riemann] and, moreover, local) probability density properties. Within a probability density support, determine all the non-intersecting non-extendable segments on which the Lebesgue integrals of a probability density vanish to additionally include the endpoints of these segments into the set of the partitioning points. Naturally, it is a Cantor set without element repetitions. Using namely the Lebesgue integral rather than the Riemann integral is especially important rather in theory than in practice, e.g. for the Dirichlet rational-number indicator function

$$I_Q(x) = 1_Q \cup 0_{R\backslash Q}$$

(Q the rational numbers, R the real numbers, R\Q the irrational numbers) whose Lebesgue integral vanishes on any real segment and the whole real axis whereas its Riemann integral does not exist on any real segment [Encyclopaedia of Mathematics].

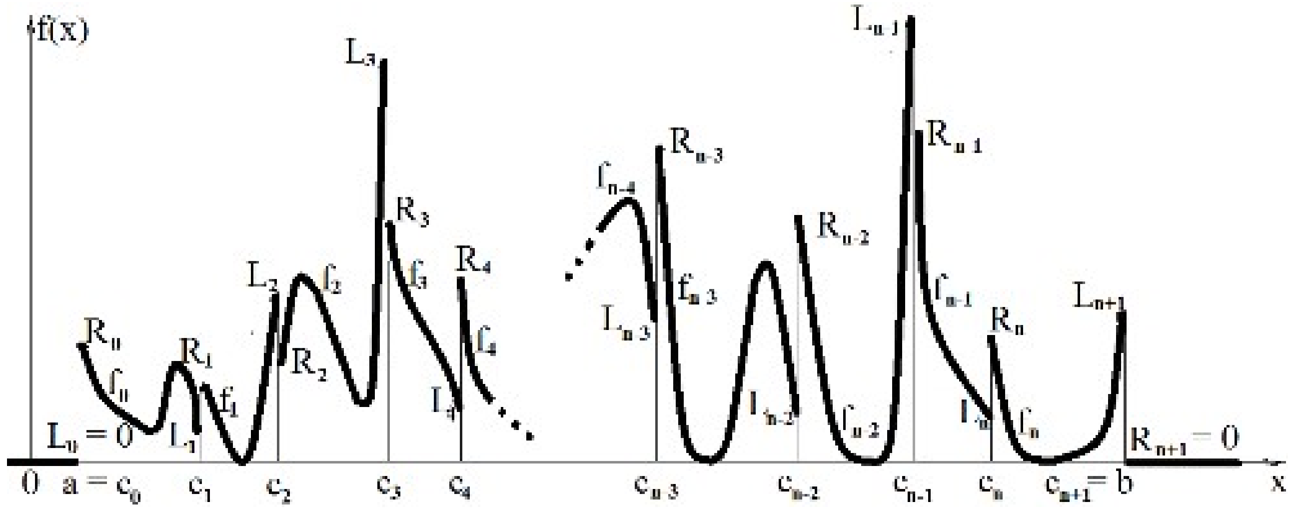

Figure 2. General one-dimensional bounded-support finite-piecewise probability density

Here probability density function $f(x)$ is as always non-negative everywhere ($-\infty < x < +\infty$) and can be positive on some so-called support which is a finite segment (closed interval)

$$-\infty < a \leq x \leq b < +\infty \ (a < b)$$

only. Let n ($n \in N = \{1, 2, ...\}$) intermediate points c_1 , c_2 , c_3 , c_4 , … , c_{n-3} , c_{n-2} , c_{n-1} , c_n in the strictly increasing order (if we need no passage to a limit to change number n of these points) so that

$$a < c_1 < c_2 < c_3 < c_4 < \ldots < c_{n-3} < c_{n-2} < c_{n-1} < c_n < b$$

divide this segment into $n + 1$ parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a ,$$
$$c_{n+1} = b ,$$
$$c(i) = c_i \ (i = 0, 1, 2, \ldots , n + 1).$$

Let us denote probability density function $f(x)$ on each of $n + 1$ open intervals

$$c_i < x < c_{i+1} \ (i = 0, 1, 2, \ldots , n)$$

as $f_i(x)$. This is suitable to separately consider these intervals, especially if probability density function $f(x)$ has on them different explicit analytical expressions.

Nota bene: A probability density function $f(x)$ with namely unit integral on D may be arbitrary. To provide integrability of probability density function $f(x)$ also multiplied by desired and/or required powers of variable x to obtain explicit (closed-form) integral (cumulative) probability distribution function $F(x)$ along with moments [Cramér, Encyclopaedia of Mathematics], use namely the simplest and most suitable classes of functions to piecewise build a desired and/or required probability density function $f(x)$ via

$$f(x) = f_i(x).$$

Among them are, e.g., some power functions including polynomials, rational, exponential, trigonometric, and hyperbolic functions, as well as their linear and nonlinear combinations. Such function variety and partition variety provide very many possibilities of solving typical classes of urgent problems also in probability theory and mathematical statistics.

At $n + 2$ points

$$c_i \ (i = 0, 1, 2, \ldots , n + 1),$$

$f(x)$ may take any finite non-negative values. The following considerations (possibly excepting mode values below) do not depend on these values. At each of $n + 2$ points

$$c_i \ (i = 0, 1, 2, \ldots , n + 1),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are any generally different finite non-negative values. Naturally, we have

$$L_0 = 0,$$
$$R_{n+1} = 0.$$

Notata bene:

1. If and only if on some of $n + 1$ open intervals

$$c_i < x < c_{i+1} \ (i = 0, 1, 2, \ldots , n),$$

probability density function $f(x)$ is namely linear, then on these intervals, the corresponding left and right one-sided limits uniquely determine

$$f(x) = R_i + (L_{i+1} - R_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= [R_i(c_{i+1} - x) + L_{i+1}(x - c_i)]/(c_{i+1} - c_i).$$

2. To provide probability density function representation unity, both on the intervals with probability density function linearity and on the intervals with its non-linearity, simply use general representation

$$f(x) = f_i(x) \mid c_i < x < c_{i+1} \ (i = 0, 1, 2, \ldots , n).$$

Using short (reduced) notation [Gelimson 2012a], represent non-negative-valued function f(x) via

$$f_{Ra\subseteq[0,\infty)}(x_{(-\infty,\infty)}) = 0_{(-\infty,c(0))\cup(c(n+1),\infty)} \cup \cup_{i=0}^{n} f_{i|Ra(i)\subseteq[0,\infty)}(x_{(c(i),c(i+1))}) \cup \cup_{i=0}^{n+1} f_{\{f(c(i))\}}(c_i),$$

or, using extended range $[0, \infty)'$ rather than ranges Ra and $Ra(i) = Ra_i$ and simplifying $f_{\{c(i)\}}(c_i)$ via identifying [Gelimson 2003a, 2003b] one-point set $\{c(i)\} = \{c_i\}$ at least here with this point $c(i) = c_i$ itself, via

$$f_{[0,\infty)'}(x_{(-\infty,\infty)}) = 0_{(-\infty,c(0))\cup(c(n+1),\infty)} \cup \cup_{i=0}^{n} f_{i|[0,\infty)'}(x_{(c(i),c(i+1))}) \cup \cup_{i=0}^{n+1} f_{f(c(i))}(c_i),$$

or, further simplifying $f_{c(i)}(c_i)$ via omitting some obvious indexes including index $f(c(i)) = f(c_i)$ coinciding with the value at argument c_i , which is admissible if and only if argument c_i is explicitly indicated, e.g. here (but NOT after replacing expression $f(c_i)$ via its value, e.g. a number), via

$$f(x) = 0_{(-\infty,c(0))\cup(c(n+1),\infty)} \cup \cup_{i=0}^{n} f_i(x_{(c(i),c(i+1))}) \cup \cup_{i=0}^{n+1} f(c_i)$$

on the whole real axis $(-\infty , \infty)$
where
index

$$Ra \subseteq [0 , \infty)$$

in

$$f_{Ra\subseteq[0,\infty)}$$

indicates the domain of dependent variable f and hence the range of function f(x);
index $(-\infty , \infty)$ in $x_{(-\infty,\infty)}$ indicates the range of independent variable x and hence the domain of one-argument function f(x);
index

$$(-\infty , c(0)) \cup (c(n+1), \infty) = (-\infty , c_0) \cup (c_{n+1} , \infty)$$

in

$$0_{(-\infty,c(0))\cup(c(n+1),\infty)}$$

indicates that function f(x) = 0 on its subdomain

$$(-\infty , c(0)) \cup (c(n+1), \infty) = (-\infty , c_0) \cup (c_{n+1} , \infty);$$

symbol $\cup$ unifies subfunctions on subdomains similarly to symbol $\cup$ in set theory and can be also indexed with an index range;
bounds 0 and n of index i in $\cup_{i=0}^{n}$ indicate that the range of index i is $\{0, 1, 2, \dots , n\}$;
index

$$(c(i), c(i+1)) = (c_i , c_{i+1})$$

in

$$x_{(c(i),c(i+1))}$$

indicates that function

$$f(x) = f_{i|Ra(i)\subseteq[0,\infty)}(x_{(c(i),c(i+1))})$$

on its subdomain

$$(c(i), c(i+1)) = (c_i , c_{i+1});$$

index

$$\{c(i)\} = \{c_i\}$$

in

$$f_{\{c(i)\}}(c_i)$$

indicates that function

$$f(x) = f(c_i)$$

on its subdomain

$$\{c(i)\} = \{c_i\}.$$

2.2. Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} dF(x) = \int_{-\infty}^{+\infty} f(x)dx = 1.$$

Nota bene: Our probability density function f(x) has real interval (a , b) as possibly extended support beyond which

$$f(x) = 0.$$

In our case we have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} f_i(x)dx .$$

Therefore, to provide a possible (an admissible) probability density function, necessary and sufficient integral normalization condition

$$\Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} f_i(x)dx = 1$$

has to be satisfied.

2.3. Normalization Transformation

Nota bene: In practice, we usually begin with a probability quasidensity function g(x) as a real-argument non-negative-valued function having a positive integral on the real axis rather than directly with a desired and/or required possible (admissible) probability density function f(x) having namely unit integral on the real axis accordingly to the normalization condition. We also in general obtain a probability quasidensity g(x) when, e.g., simply adding some probability densities together or, more general, building a linear combination (whose factors sum is not 1) of them.

Let a probability quasidensity g(x) be a real-argument non-negative-valued function

$$g(x) = 0_{(-\infty\,,\,c(0))\,\cup\,(c(n+1),\,\infty)} \cup \cup_{i=0}^{n} g_i(x_{(c(i),\,c(i+1))}) \cup \cup_{i=0}^{n+1} g(c_i)$$

similar to a desired and/or required possible (admissible) probability density

$$f(x) = 0_{(-\infty\,,\,c(0))\,\cup\,(c(n+1),\,\infty)} \cup \cup_{i=0}^{n} f_i(x_{(c(i),\,c(i+1))}) \cup \cup_{i=0}^{n+1} f(c_i)$$

but having any namely positive but not obligatorily unit integral

$$\int_{-\infty}^{+\infty} g(x)dx = \int_a^b g(x)dx$$

$$= \Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} g_i(x)dx > 0$$

on the real axis. This positivity makes it possible to simply apply the typical straightforward poinwise proportionality transformation idea to obtain probability density f(x) via dividing probability quasidensity g(x) by its integral:

$$f(x) = g(x)/\int_{-\infty}^{+\infty} g(x)dx$$

$$= 0_{(-\infty\,,\,c(0))\,\cup\,(c(n+1),\,\infty)} \cup \cup_{i=0}^{n} g_i(x_{(c(i),\,c(i+1))})/\Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} g_i(x)dx \cup \cup_{i=0}^{n+1} g(c_i)/\Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} g_i(x)dx\ .$$

2.4. Integral (Cumulative) Probability Distribution Function

Integral (cumulative) probability distribution function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

is probability $P(X \leq x)$ that real-number random variable X takes a real-number value not greater than x .

Nota bene: Our probability density function $f(x)$ has real interval (a , b) as possibly extended support beyond which

$$f(x) = 0.$$

For $x < a = c_0$, this definition gives

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = 0.$$

For $x \geq b = c_{n+1}$, this definition gives

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = 1.$$

Otherwise, namely for

$$x \mid c_0 = a \leq x < b = c_{n+1} ,$$

we can use the following natural idea, way, and algorithm:

1. Determine such value j of index i that

$$c_j \leq x < c_{j+1} .$$

There exists such value j and namely the only. Indeed, consider set

$$\{i \mid i \in \{0, 1, 2, ... , n\}, c_i \leq x\}.$$

It is non-empty because it contains at least 0 for which

$$c_0 = a < x$$

and hence

$$c_0 \leq x .$$

It is finite and strictly ordered by relation < . Therefore, there exists its maximal element

$$j = \max\{i \mid i \in \{0, 1, 2, ... , n\}, c_i \leq x\},$$

and this maximal element is namely the only. Then for this maximal element j , in addition to

$$c_j \leq x ,$$

inequality

$$x < c_{j+1}$$

also holds. Indeed, otherwise, we would have

$$x \geq c_{j+1}$$

and

$$c_{j+1} \leq x ,$$

so that this j could not be namely the maximal element of this set.

2. Determine

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f_j(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f_j(t)dt .$$

Notata bene:

1. Unfortunately, the last representation does not hold by

$$x > b = c_{n+1}$$

because generally

$$f(t_{(c(n+1), x)}) = f_{n+1}(t) \equiv 0 \not\equiv f_n(t).$$

Therefore, to provide generality, we have to use $f(t)$ rather than $f_j(t)$ in the two last integrals. Namely,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt .$$

2. For $x < a = c_0$, we have empty set

$$\{i \mid i \in \{0, 1, 2, ... , n\}, c_i \le x\} = \varnothing .$$

Hence there is no such j (or we may consider that j is the empty element:

$$j = \#).$$

Then we may also consider

$$\Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt = 0$$

as the empty sum, as well as

$$\int_{c(j)}^{x} f_j(t)dt = 0$$

as the empty integral which also has the nature of a sum, and therefore

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_a^x f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt = 0,$$

which is correct.

3. For $x \ge b = c_{n+1}$, we have empty set

$$j = \max\{i \mid i \in \{0, 1, 2, ... , n\}, c_i \le x\} = n .$$

Then we have

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_a^x f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{n-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(n)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{n-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(n)}^{c(n+1)} f_n(t)dt + \int_{c(n+1)}^{x} f_{n+1}(t)dt$$
$$= \Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(n+1)}^{x} f_{n+1}(t)dt$$
$$= \int_a^b f(t)dt + \int_{c(n+1)}^{x} 0dt = 1,$$

which is correct.

Therefore, obtained expression

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_a^x f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt$$

for integral (cumulative) probability distribution function $F(x)$ is valid at any real x .

Nota bene: Alternatively, using namely the supremum inverse function

$$z = \underline{c}|(x),$$

simply determine

$$j = \lfloor \underline{c}|(x) \rfloor.$$

Here the floor (or entier) function [Encyclopaedia of Mathematics]

$$v = \lfloor w \rfloor$$

of real argument w gives the largest integer v less than or equal to w .

Then integral (cumulative) probability distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_a^x f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{\lfloor \underline{c}|(x) \rfloor-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(\lfloor \underline{c}|(x) \rfloor)}^{x} f(t)dt .$$

Notata bene:

1. For $x < a = c_0$, we have the empty elements

$$\underline{c}|(x) = \#$$

and

$$j = \lfloor \underline{c}|(x) \rfloor = \# .$$

Then we may consider

$$\Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt = 0$$

as the empty sum, as well as

$$\int_{c(j)}^{x} f(t)dt = 0$$

as the empty integral which also has the nature of a sum, and therefore

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{\lfloor\underline{c}|(x)|-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(\lfloor\underline{c}|(x)\rfloor)}^{x} f(t)dt = 0,$$

which is correct.

2. For $x \ge b = c_{n+1}$, we have empty set

$$j = \lfloor\underline{c}|(x)\rfloor = n .$$

Then we have

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{\lfloor\underline{c}|(x)|-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(\lfloor\underline{c}|(x)\rfloor)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{n-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(n)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{n-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(n)}^{c(n+1)} f_n(t)dt + \int_{c(n+1)}^{x} f_{n+1}(t)dt$$
$$= \Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(n+1)}^{x} f_{n+1}(t)dt$$
$$= \int_{a}^{b} f(t)dt + \int_{c(n+1)}^{x} 0dt = 1,$$

which is correct.

Therefore, obtained expression

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{\lfloor\underline{c}|(x)|-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(\lfloor\underline{c}|(x)\rfloor)}^{x} f(t)dt$$

for integral (cumulative) probability distribution function F(x) is valid at any real x .

2.5. Integral (Cumulative) Probability Distribution Function Inversion

To begin with, note that the general consideration for a probability density holds. Namely, integral (cumulative) probability distribution function $F(x)$ has domain

$$D = R = (-\infty , \infty)$$

and range $Ra = [0, 1]$. $F(x)$ is a one-argument one-value real-number function. It can be not only strictly monotonically increasing, but also locally non-strictly monotonically increasing. If its arbitrary image y belonging to range $Ra = [0, 1]$ has at least two distinct preimages x_1 and x_2 , then there is an open interval, one of two half-open and half-closed intervals, or a segment as a closed interval

$$(|x , x|), (|x , x|], [|x , x|), [|x , x|],$$

or

$$(|\underline{F}(y), \underline{F}|(y)), (|\underline{F}(y), \underline{F}|(y)], [|\underline{F}(y), \underline{F}|(y)), [|\underline{F}(y), \underline{F}|(y)],$$

whose either excluded or included (independently from one another) endpoints are

$$|x = |\underline{F}(y)$$

and

$$x| = \underline{F}|(y)$$

and which may be regarded as the total preimage of image y .

Nota bene: On this interval possibly excluding its subset of zero measure, probability density function $f(x)$ vanishes. Otherwise, integral (cumulative) probability distribution function $y = F(x)$ could not be constant on this interval.

Therefore, there exist the infimum inverse function

$$x = |\underline{F}(y)$$

and the supremum inverse function

$$x = \underline{F}|(y).$$

These one-argument one-value real-number functions are namely the both extreme functions among all the generally many-valued functions inverse to generally non-strictly monotonically increasing one-argument one-value real-number function

$$y = F(x).$$

Further, in our case of a piecewise probability density, we obtained for integral (cumulative) probability distribution function $F(x)$ the following explicit expression:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} f(t)dt = \int_{c(0)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt$$
$$= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(j)}^{x} f_j(t)dt$$
$$= \Sigma_{i=0}^{[\underline{c}|(x)]-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c([\underline{c}|(x)])}^{x} f_{[c|(x)]}(t)dt .$$

Within a probability density support, we have already determined all the non-intersecting non-extendable segments on which the Lebesgue integrals of a probability density vanish to additionally include the endpoints of these segments into the set of the partitioning points.

Nota bene: There can exist isolated absolute antimodes at any of which a probability density vanishes whereas there is an antimode neighborhood (interval on which this antimode is namely an internal point) so that at any other (than this antimode) point, a probability density is strictly positive. Then the Lebesgue integral of a probability density on any interval containing such an isolated absolute antimode is strictly positive, too. If the analytical expression of a probability density function does not change at such an antimode, then there is no reason to additionally include the endpoints of these segments into the set of the partitioning points. Moreover, an isolated absolute antimode has no influence on the qualitative behavior of integral (cumulative) probability distribution function inversion at all.

We may also regard the both external (with respect to the extended support

$$S' = [a , b]$$

of a probability density) infinite half-closed intervals

$$(-\infty , a]$$

and

$$[b , \infty)$$

on which on which the Lebesgue integrals of a probability density vanish to be further non-extendable. Otherwise, simply maximally extend these infinite half-closed intervals.

Notata bene:

1. The both finite endpoints a and b are hence already included as shown. Intersecting these infinite half-closed intervals with the extended support S' = [a , b] at these endpoints a and b brings no problem.

2. The non-extendability of these both external infinite half-closed intervals $(-\infty , a]$ and $[b , \infty)$ provides the external non-reducibility of the extended support S' = [a , b], and vice versa.

3. Also an externally non-reducible support S' = [a , b] can remain internally extended because it may include internal segments on which the Lebesgue integrals of a probability density vanish.

Namely, besides the both external infinite half-closed intervals $(-\infty , a]$ and $[b , \infty)$, there can be a finite set of non-intersecting internal (with respect to the externally non-reducible possibly internally extended support S' = [a , b]) further non-extendable finite segments

$$[s_k , t_k] \subset S' = [a , b] \mid k = 1, 2, 3, \ldots , K$$

so denoted in the increasing order that

$$a < s_1 < t_1 < s_2 < t_2 < s_3 < t_3 < \ldots < s_K < t_K < b \, .$$

Nota bene: Each segment $[s_k , t_k]$ is namely a proper subset of support S' = [a , b]. Otherwise, the Lebesgue integral of a probability density on the whole real axis would vanish whereas its value is namely unit.

Naturally, the set of all the segments endpoints is namely a proper subset of the set of all the partitioning points:

$$\cup_{k=1}^{K}\{s_k , t_k\} \subset \cup_{i=0}^{n+1}\{c_i\}.$$

But to simpler show the nature and essence of inverting integral (cumulative) probability distribution function y = F(x), it seems to be suitable to introduce these new designations rather than to select some of these points c_i especially with building their pairs.

To provide designation unity, additionally denote

$$t_0 = a$$

and

$$s_{K+1} = b \, .$$

Then we have

$$t_0 < s_1 < t_1 < s_2 < t_2 < s_3 < t_3 < \ldots < s_K < t_K < s_{K+1} \, .$$

Nota bene: Along with set

$$(-\infty , a] \cup \cup_{k=1}^{K}[s_k , t_k] \cup [b , \infty)$$

of the non-intersecting external (with respect to extended support S' = [a , b]) half-closed infinite intervals and internal finite segments so that on each subset of this set, the Lebesgue integral of a probability density vanishes, we have quasicomplementary set

$$\cup_{k=1}^{K}[t_k , s_{k+1}]$$

of the non-intersecting internal (with respect to extended support S' = [a , b]) finite segments so that on each nonzero-measure subset of this set, the Lebesgue integral of a probability density is strictly positive.

Now return to n ($n \in N = \{1, 2, ...\}$) intermediate points $c_1 , c_2 , c_3 , c_4 , \ldots , c_{n-3} , c_{n-2} , c_{n-1} , c_n$ in the strictly increasing order (also here we need no passage to a limit to change number n of these points) so that

$$a = c_0 < c_1 < c_2 < c_3 < c_4 < \ldots < c_{n-3} < c_{n-2} < c_{n-1} < c_n < c_{n+1} = b$$

which divide this segment into n + 1 parts (pieces) of generally different lengths.

Nota bene: On any segment in

$$\cup_{k=1}^{K}[s_k , t_k],$$

there is no internal point c_i because on each nonzero-measure subset of this segment, the Lebesgue integral of probability density f(x) vanishes, possible nonzero values of f(x) on a zero-measure subset of this segment have no influence on this integral and on probability itself, so we may consider that such a probability density identically vanishes on this segment.

Therefore, for any index

$$k = 1, 2, 3, \ldots , K ,$$

there is such index

$$i(k) \in \{1, 2, ... , n - 1\}$$

depending on k that

$$c_{i(k)} = s_k$$

and

$$c_{i(k)+1} = t_k .$$

Nota bene: i = 0 or i = n would give

$$s_k = c_0 = a$$

or

$$t_k = c_{n+1} = b ,$$

respectively, which both is impossible.

Then the increase behavior of integral (cumulative) probability distribution function

$$y_{[0, 1]} = F_{[0, 1]}(x_{(-\infty , \infty)}),$$

or simply

$$y = F(x),$$

is in detail as follows.

On external infinite half-closed interval $(-\infty , a]$, this function identically vanishes:

$$y = F(x) \equiv 0.$$

On internal segment $[c_0 , c_{i(1)}] = [a , c_{i(1)}]$, this function

$$y = F(x)$$

strictly monotonically increases with taking the following values at partitioning points c_i :

$$F(c_0) = F(a) = 0;$$

$$F(c_1) = \int_{c(0)}^{c(1)} f_0(t)dt ;$$

$$F(c_2) = \int_{c(0)}^{c(1)} f_0(t)dt + \int_{c(1)}^{c(2)} f_1(t)dt ;$$

$$F(c_3) = \int_{c(0)}^{c(1)} f_0(t)dt + \int_{c(1)}^{c(2)} f_1(t)dt + \int_{c(2)}^{c(3)} f_2(t)dt ;$$

....

$$F(c_{i(1)}) = \int_{c(0)}^{c(1)} f_0(t)dt + \int_{c(1)}^{c(2)} f_1(t)dt + \int_{c(2)}^{c(3)} f_2(t)dt + ... + \int_{c(i(1)-1)}^{c(i(1))} f_{i(1)-1}(t)dt = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt .$$

On internal segment $[c_{i(1)} , c_{i(1)+1}]$, this function is identical constant:

$$y = F(x) \equiv \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt .$$

On internal segment $[c_{i(1)+1} , c_{i(2)}]$, this function

$$y = F(x)$$

strictly monotonically increases with taking the following values at partitioning points c_i :

$$F(c_{i(1)+1}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt ;$$

$$F(c_{i(1)+2}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(i(1)+1)}^{c(i(1)+2)} f_{i(1)+1}(t)dt ;$$

$$F(c_{i(1)+3}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(i(1)+1)}^{c(i(1)+2)} f_{i(1)+1}(t)dt + \int_{c(i(1)+2)}^{c(i(1)+3)} f_{i(1)+2}(t)dt ;$$

....

$$F(c_{i(2)}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \Sigma_{i=i(1)+1}^{i(2)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt .$$

On internal segment $[c_{i(2)} , c_{i(2)+1}]$, this function is identical constant:

$$y = F(x) \equiv \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \Sigma_{i=i(1)+1}^{i(2)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt \equiv [\Sigma_{i=0}^{i(1)-1} + \Sigma_{i=i(1)+1}^{i(2)-1}] \int_{c(i)}^{c(i+1)} f_i(t)dt .$$

....

....

....

On internal segment $[c_{i(K)} , c_{i(K)+1}]$, this function is identical constant:

$$y = F(x) \equiv [\Sigma_{i=0}{}^{i(1)-1} + \Sigma_{i=i(1)+1}{}^{i(2)-1} + \ldots + \Sigma_{i=i(K-1)+1}{}^{i(K)-1}] \int_{c(i)}{}^{c(i+1)} f_i(t)dt$$
$$\equiv \Sigma_{i=0,\ 1,\ 2,\ \ldots\ ,\ i(1)-1;\ \#\ ;\ i(1)+1,\ i(1)+2,\ i(1)+3,\ \ldots\ ,\ i(2)-1;\ \#\ ;\ \ldots\ ;\ i(K-1)+1,\ i(K-1)+2,\ i(K-1)+3,\ \ldots\ ,\ i(K)-1;\ \#} \int_{c(i)}{}^{c(i+1)} f_i(t)dt$$
$$\equiv \Sigma_{i\in\{0,\ 1,\ 2,\ \ldots\ ,\ i(K)\}\backslash\{i(1),\ i(2),\ \ldots\ ,\ i(K)\}} \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ .$$

On internal segment $[c_{i(K)+1}\ ,\ c_{n+1}] = [c_{i(K)+1}\ ,\ b]$, this function
$$y = F(x)$$
strictly monotonically increases with taking the following values at partitioning points c_i :
$$F(c_{i(K)+1}) = \Sigma_{i\in\{0,\ 1,\ 2,\ \ldots\ ,\ i(K)\}\backslash\{i(1),\ i(2),\ \ldots\ ,\ i(K)\}} \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ ;$$
$$F(c_{i(K)+2}) = \Sigma_{i\in\{0,\ 1,\ 2,\ \ldots\ ,\ i(K)+1\}\backslash\{i(1),\ i(2),\ \ldots\ ,\ i(K)\}} \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ ;$$
$$F(c_{i(K)+3}) = \Sigma_{i\in\{0,\ 1,\ 2,\ \ldots\ ,\ i(K)+2\}\backslash\{i(1),\ i(2),\ \ldots\ ,\ i(K)\}} \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ ;\ ;$$
....
$$F(c_{n+1}) = \Sigma_{i\in\{0,\ 1,\ 2,\ \ldots\ ,\ n\}\backslash\{i(1),\ i(2),\ \ldots\ ,\ i(K)\}} \int_{c(i)}{}^{c(i+1)} f_i(t)dt = 1.$$

On external infinite half-closed interval $[b\ ,\ \infty)$, this function is identical unit:
$$y = F(x) \equiv 1.$$
Then inverting integral (cumulative) probability distribution function
$$y_{[0,\ 1]} = F_{[0,\ 1]}(x_{(-\infty\ ,\ \infty)})$$
via inverse function
$$x_{(-\infty\ ,\ \infty)} = \underline{F}_{(-\infty\ ,\ \infty)}(y_{[0,\ 1]})$$
gives the following.
If
$$y = 0,$$
then
$$x = \underline{F}(0) = (-\infty\ ,\ a].$$
If
$$y = \Sigma_{i=0}{}^{i(1)-1} \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ ,$$
then
$$x = \underline{F}(\Sigma_{i=0}{}^{i(1)-1} \int_{c(i)}{}^{c(i+1)} f_i(t)dt) = [c_{i(1)}\ ,\ c_{i(1)+1}].$$
If
$$y = [\Sigma_{i=0}{}^{i(1)-1} + \Sigma_{i=i(1)+1}{}^{i(2)-1}] \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ ,$$
then
$$x = \underline{F}([\Sigma_{i=0}{}^{i(1)-1} + \Sigma_{i=i(1)+1}{}^{i(2)-1}] \int_{c(i)}{}^{c(i+1)} f_i(t)dt) = [c_{i(2)}\ ,\ c_{i(2)+1}].$$
....

If
$$y = \Sigma_{i\in\{0,\ 1,\ 2,\ \ldots\ ,\ i(K)\}\backslash\{i(1),\ i(2),\ \ldots\ ,\ i(K)\}} \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ ,$$
then
$$x = \underline{F}(\Sigma_{i\in\{0,\ 1,\ 2,\ \ldots\ ,\ i(K)\}\backslash\{i(1),\ i(2),\ \ldots\ ,\ i(K)\}} \int_{c(i)}{}^{c(i+1)} f_i(t)dt) = [c_{i(K)}\ ,\ c_{i(K)+1}].$$
If
$$y = 1,$$
then
$$x = \underline{F}(1) = [b\ ,\ \infty).$$
For other
$$y \in [0,\ 1],$$
integral (cumulative) probability distribution function
$$y_{[0,\ 1]} = F_{[0,\ 1]}(x_{(-\infty\ ,\ \infty)})$$
strictly monotonically increases. Therefore, its inverse function
$$x_{(-\infty\ ,\ \infty)} = \underline{F}_{(-\infty\ ,\ \infty)}(y_{[0,\ 1]})$$
returns exactly one value x . To find it, apply, e.g., the following natural algorithm:
1. Determine the greatest of the above consequent values of y (beginning with 0 and ending with 1) which is still less than the given value of y . Let this greatest value of y be, e.g.,
$$y = \Sigma_{i=0}{}^{i(1)-1} \int_{c(i)}{}^{c(i+1)} f_i(t)dt\ .$$
2. Also consider in the same sequence the next value of y which is naturally already greater than the

given value of y . In our example, this next value of y is

$$y = [\Sigma_{i=0}^{i(1)-1} + \Sigma_{i=i(1)+1}^{i(2)-1}] \int_{c(i)}^{c(i+1)} f_i(t)dt .$$

3. Consider the sequence of the internal segments $[c_i , c_{i+1}]$ on which integral (cumulative) probability distribution function

$$y_{[0, 1]} = F_{[0, 1]}(x_{(-\infty , \infty)})$$

strictly monotonically increases and which provide both the last two values and the intermediate values of y . In our example, this sequence and these values of y are as follows:
On internal segment $[c_{i(1)+1} , c_{i(2)}]$, this function

$$y = F(x)$$

strictly monotonically increases with taking the following values at partitioning points c_i :

$$F(c_{i(1)+1}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt ;$$

$$F(c_{i(1)+2}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(i(1)+1)}^{c(i(1)+2)} f_{i(1)+1}(t)dt ;$$

$$F(c_{i(1)+3}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(i(1)+1)}^{c(i(1)+2)} f_{i(1)+1}(t)dt + \int_{c(i(1)+2)}^{c(i(1)+3)} f_{i(1)+2}(t)dt ;$$

....

$$F(c_{i(2)}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \Sigma_{i=i(1)+1}^{i(2)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt \equiv [\Sigma_{i=0}^{i(1)-1} + \Sigma_{i=i(1)+1}^{i(2)-1}] \int_{c(i)}^{c(i+1)} f_i(t)dt .$$

4. Determine the greatest of these consequent values of y which is still not greater than the given value of y . Let this greatest value of y be, e.g.,

$$y = F(c_{i(1)+2}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(i(1)+1)}^{c(i(1)+2)} f_{i(1)+1}(t)dt .$$

5. Also consider in the same sequence the next value of y which is naturally already greater than the given value of y . In our example, this next value of y is

$$y = F(c_{i(1)+3}) = \Sigma_{i=0}^{i(1)-1} \int_{c(i)}^{c(i+1)} f_i(t)dt + \int_{c(i(1)+1)}^{c(i(1)+2)} f_{i(1)+1}(t)dt + \int_{c(i(1)+2)}^{c(i(1)+3)} f_{i(1)+2}(t)dt .$$

6. Now for the given value of y , we have

$$F(c_{i(1)+2}) \le y < F(c_{i(1)+3})$$

and hence

$$c_{i(1)+2} \le x < c_{i(1)+3} .$$

7. If

$$y = F(c_{i(1)+2}),$$

then desired and required

$$x < c_{i(1)+2} .$$

8. If

$$F(c_{i(1)+2}) < y < F(c_{i(1)+3}),$$

then

$$c_{i(1)+2} < x < c_{i(1)+3} .$$

9. To determine

$$x = \underline{F}(y),$$

solve equation

$$\int_{c(i(1)+2)}^{x} f_{i(1)+2}(t)dt = y - F(c_{i(1)+2})$$

in x , which has exactly one solution on interval

$$c_{i(1)+2} < x < c_{i(1)+3} .$$

Notata bene:
1. Such complicated indexes correspond to our example only. We may simply replace i(1) + 2 via i , and it is even much more general. Then to determine

$$x = \underline{F}(y),$$

solve equation

$$\int_{c(i)}^{x} f_i(t)dt = y - F(c_i)$$

in x , which has exactly one solution on interval

$$c_i < x < c_{i+1}$$

with

$$F(c_i) < y < F(c_{i+1})$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c_i < x < c_{i+1} .$$

2. To further generalize this problem, simply omit index i at c_i and f_i with replacing c_{i+1} via d . Then to determine

$$x = \underline{F}(y),$$

solve equation

$$\int_c^x f(t)dt = y - F(c)$$

in x , which has exactly one solution on interval

$$c < x < d$$

with

$$F(c) < y < F(d)$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c < x < d .$$

3. Even if probability density f(x) or here f(t) is (indefinitely) integrable, this equation in x is explicitly closed-form resolvable in some particular cases only.
4. If probability density f(x) is strictly positive (because of strictly monotonically increasing F(x)) constant C on interval

$$c < x < d ,$$

then we solve equation

$$\int_c^x Cdt = C(x - c) = y - F(c)$$

in x , which has exactly one solution

$$x = c + [y - F(c)]/C$$

on interval

$$c < x < d$$

with

$$F(c) < y < F(d)$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c < x < d .$$

Indeed, we have

$$x = c + [y - F(c)]/C > c$$

because

$$y - F(c) > 0;$$

$$d - x = d - c - [y - F(c)]/C > d - c - [F(d) - F(c)]/C > 0$$

because

$$F(x) = Cx$$

with possibly adding some indefinite integration constant is strictly monotonically increasing.
5. If probability density f(x) is a non-negative polynomial of degree 1, 2, or 3 on interval

$$c < x < d ,$$

then indefinite integration in equation

$$\int_c^x f(t)dt = y - F(c)$$

in x provides adding 1 to this degree and we have to solve the corresponding quadratic, cubic, or quartic equation, respectively, which has exactly one solution on interval

$$c < x < d$$

with

$$F(c) < y < F(d)$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c < x < d .$$

6. If probability density is a power function with non-negative variable base of any positive degree, namely

$$f(x) = C(x - c)^D$$

with constants

$$C > 0, D > 0$$

on interval

$$c < x < d ,$$

then indefinite integration in equation

$$\int_c^x f(t)dt = y - F(c)$$

in x gives

$$\int_c^x C(t - c)^D dt = y - F(c),$$
$$C(x - c)^{D+1}/(D + 1) = y - F(c),$$
$$(x - c)^{D+1} = [y - F(c)](D + 1)/C$$

which has exactly one solution

$$x = c + \{[y - F(c)](D + 1)/C\}^{1/(D+1)}$$

on interval

$$c < x < d$$

with

$$F(c) < y < F(d)$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c < x < d .$$

7. If probability density is an exponential function with a linear exponent, namely

$$f(x) = Ce^{Dx}$$

with constants

$$C > 0$$

and

$$D \neq 0$$

on interval

$$c < x < d ,$$

then indefinite integration in equation

$$\int_c^x f(t)dt = y - F(c)$$

in x gives

$$\int_c^x Ce^{Dt}dt = y - F(c),$$
$$C(e^{Dx} - e^{Dc})/D = y - F(c),$$
$$e^{Dx} = e^{Dc} + [y - F(c)]D/C ,$$

which has exactly one solution

$$x = D^{-1} \ln\{e^{Dc} + [y - F(c)]D/C\}$$

on interval

$$c < x < d$$

with

$$F(c) < y < F(d)$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c < x < d .$$

8. If probability density is a trigonometric function non-negative on interval

$$c < x < d ,$$

e.g.

$$f(x) = C\sin[\pi(x - c)/(d - c)]$$

with constant

$$C > 0,$$

then indefinite integration in equation

$$\int_c^x f(t)dt = y - F(c)$$

in x gives

$$\int_c^x C\sin[\pi(t - c)/(d - c)]dt = y - F(c),$$
$$C(d - c)/\pi \{1 - \cos[\pi(x - c)/(d - c)]\} = y - F(c),$$
$$1 - \cos[\pi(x - c)/(d - c)] = \pi[y - F(c)]/[C(d - c)],$$
$$\cos[\pi(x - c)/(d - c)] = 1 - \pi[y - F(c)]/[C(d - c)],$$
$$\pi(x - c)/(d - c) = \arccos\{1 - \pi[y - F(c)]/[C(d - c)]\},$$

which has exactly one solution

$$x = c + (d - c)/\pi \arccos\{1 - \pi[y - F(c)]/[C(d - c)]\}$$

on interval

$$c < x < d$$

with

$$F(c) < y < F(d)$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c < x < d .$$

9. Generally, even if probability density may be indefinitely non-integrable at all, we can completely avoid indefinite integration. Consider such a typical general mathematical (not only probabilistic) problem with a namely positive integrand function (which may vanish on a zero-measure subset of the integration interval) and hence with an integral function strictly monotonically increasing on this interval. Use the following natural and direct integral sum accumulation method. To begin with, take any positive desired and/or required precision $\Delta > 0$ of determining the desired value of an independent variable on an integration interval of length L which we may consider half-closed from the left for definiteness. If necessary, reduce this precision to provide the namely integer divisionability of the integration interval length by this precision so that their ratio is a natural number which we denote as n :

$$n = L/\Delta ,$$
$$L = n\Delta .$$

Partition this integration interval into n parts also half-closed from the left all of length

$$\Delta = L/n .$$

Multiply the desired and/or required integral value V with this precision Δ to obtain their product $V\Delta$. Using the integrand function monotonicity properties, determine both the maximal value and the minimal value of this function on every of these n parts. Simply incrementally add these part maximums beginning with the left endpoint of the integration interval product up to exceeding product $V\Delta$.

$$y = F(x)$$

on interval f(x)
$c < x < d$ via a trigonometric function non-negative on interval

$$c < x < d ,$$

e.g.

$$f(x) = C\sin[\pi(x - c)/(d - c)]$$

with constant

$$C > 0,$$

then in equation

$$\int_c^x f(t)dt = y - F(c)$$

in x gives

$$\int_c^x C\sin[\pi(t - c)/(d - c)]dt = y - F(c),$$
$$C(d - c)/\pi \{1 - \cos[\pi(x - c)/(d - c)]\} = y - F(c),$$
$$1 - \cos[\pi(x - c)/(d - c)] = \pi[y - F(c)]/[C(d - c)],$$
$$\cos[\pi(x - c)/(d - c)] = 1 - \pi[y - F(c)]/[C(d - c)],$$
$$\pi(x - c)/(d - c) = \arccos\{1 - \pi[y - F(c)]/[C(d - c)]\},$$

which has exactly one solution

$$x = c + (d - c)/\pi \arccos\{1 - \pi[y - F(c)]/[C(d - c)]\}$$

on interval

$$c < x < d$$

with

$$F(c) < y < F(d)$$

and strictly monotonically increasing

$$y = F(x)$$

on interval

$$c < x < d .$$

10.

2.5. Mean Value (Mathematical Expectation)

Use the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx\ .$$

In our case we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} xf(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [R_i(c_{i+1}x - x^2) + L_{i+1}(x^2 - c_ix)]/(c_{i+1} - c_i)\ dx$$
$$= \Sigma_{i=0}^n \{R_i[c_{i+1}(c_{i+1}^2 - c_i^2)/2 - (c_{i+1}^3 - c_i^3)/3] + L_{i+1}[(c_{i+1}^3 - c_i^3)/3 - c_i(c_{i+1}^2 - c_i^2)/2]/(c_{i+1} - c_i)\}$$
$$= 1/6\ \Sigma_{i=0}^n \{R_i[3c_{i+1}(c_{i+1} + c_i) - 2(c_{i+1}^2 + c_{i+1}c_i + c_i^2)] + L_{i+1}[2(c_{i+1}^2 + c_{i+1}c_i + c_i^2) - 3c_i(c_{i+1} + c_i)]\}$$
$$= 1/6\ \Sigma_{i=0}^n [R_i(c_{i+1}^2 + c_{i+1}c_i - 2c_i^2) + L_{i+1}(2c_{i+1}^2 - c_{i+1}c_i - c_i^2)]$$
$$= 1/6\ \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1} + 2c_i) + L_{i+1}(2c_{i+1} + c_i)]$$

and, finally,

$$\mu = \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(2c_i + c_{i+1}) + L_{i+1}(c_i + 2c_{i+1})]/6.$$

2.6. Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \le \nu) \ge 1/2$$

and

$$P(X \ge \nu) \ge 1/2.$$

For a continual real-number random variable X ,

$$P(X \le \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \ge \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the following natural idea, way, and algorithm:

1. First consider

$$c_i \ (i = 0, 1, 2, \dots , n + 1)$$

not far from μ and determine both

$$L = \max\{i \mid \int_{-\infty}^{c(i)} f(x)dx < 1/2\}$$

and

$$R = \min\{i \mid \int_{c(i)}^{+\infty} f(x)dx < 1/2\}.$$

Then both

$$\int_{-\infty}^{c(L+1)} f(x)dx \ge 1/2$$

and

$$\int_{c(R-1)}^{+\infty} f(x)dx \ge 1/2.$$

2. On half-closed interval

$$c(L) = c_L < \nu \le c_{L+1} = c(L+1),$$

determine

$$\nu_{min} = \inf\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}.$$

3. On half-closed interval

$$c(R-1) = c_{R-1} \le \nu < c_R = c(R),$$

determine

$$\nu_{max} = \sup\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\}.$$

4. Then the set of all the median values ν is the interval whose endpoints are

$$\nu_{min} \le \nu_{max}$$

each of which is included into the interval if and only if the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and/or

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

respectively.

Notata bene:

1. If

$$\nu_{min} = \nu_{max} ,$$

then the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

hence the closed interval

$$\nu_{min} \le \nu \le \nu_{max}$$

contains the only median value

$$\nu = \nu_{min} = \nu_{max} .$$

2. If

$$\nu_{min} < \nu_{max} ,$$

then the integral of f(x) on the interval whose endpoints are ν_{min} and ν_{max} vanishes independently of their including or excluding. Hence on this interval, non-negative probability density function f(x) also vanishes possibly excepting points whose set has zero measure (in our case, a finite set).

2.7. Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density function $f(x)$ takes its maximum value f_{max} . For a continual probability density function, generalize this definition in the following directions:
1. Replace the maximum value f_{max} with the supremum value f_{sup} which always exists. The reason is that it is possible (for a piecewise linear probability density, too) that function $f(x)$ is discontinuous and does not take the supremum value f_{sup} so that the maximum value f_{max} does not exist at all.
2. Extend the range of function $f(x)$, i.e. the set of values function $f(x)$ really (truly) takes, via all the limiting points of this set. Then the extended range is a closed set and contains, in particular, the supremum value f_{sup} .
3. Extend the domain of function $f(x)$, i.e. the set of points at which function $f(x)$ is properly defined, via all the limiting points of this set. Then the extended domain is a closed set which contains all its limiting points.
4. Admit modes to also correspond to the one-sided limits of function $f(x)$ separately if necessary. This is important for discontinuous function $f(x)$ with jumps.
5. At any interval endpoint c_i , along with the given value of $f(c_i)$, take into account the one-sided limits L_i and R_i of function $f(x)$, e.g. any of the following reasonable options for value $f(c_i)$:
5.1. Take the given value of $f(c_i)$ itself.
5.2. Take

$$f(c_i) = \max\{L_i , R_i\}.$$

5.3. Take

$$f(c_i) = (L_i + R_i)/2.$$

6. At any interval endpoint c_i , along with c_i itself, take into account the one-sided limiting points c_i - 0 and $c_i + 0$ corresponding to one-sided limits L_i and R_i of function $f(x)$, respectively, e.g. any of the following reasonable options for c_i :
6.1. Take the given value of c_i itself.
6.2. For modes, rather than c_i , consider

$$c_i - 0 \text{ if } L_i > R_i ,$$
$$c_i + 0 \text{ if } L_i < R_i ,$$

and quantiset [Gelimson 2003a, 2003b]

$$\{{}_{1/2}(c_i - 0), {}_{1/2}(c_i + 0)\}^\circ \text{ if } L_i = R_i .$$

This quantiset consists of two quantielements

$${}_{1/2}(c_i - 0), {}_{1/2}(c_i + 0)$$

with bases

$$c_i - 0, c_i + 0,$$

respectively.
Here each of elements $c_i - 0$ and $c_i + 0$ has quantity 1/2 so that the total unit quantity is equally divided between these both elements.
In particular, for a piecewise linear probability density function $f(x)$, anyone of the following values can reasonably play the role of f_{sup} :

$$\max\{\max\{f(c_i) \mid i = 0, 1, 2, \dots , n + 1\}, \max\{L_i \mid i = 0, 1, \dots , n + 1\}, \max\{R_i \mid i = 0, 1, \dots , n + 1\}\},$$
$$\max\{\max\{f(c_i) \mid i = 0, 1, 2, \dots , n + 1\}, \max\{(L_i + R_i)/2 \mid i = 0, 1, 2, \dots , n + 1\}\},$$
$$\max\{\max\{L_i \mid i = 0, 1, 2, \dots , n + 1\}, \max\{R_i \mid i = 0, 1, 2, \dots , n + 1\}\},$$
$$\max\{(L_i + R_i)/2 \mid i = 0, 1, 2, \dots , n + 1\}.$$

If $f(c_i) = f_{sup}$ at some i , then c_i at this i is one of the modes.
If $L_i = f_{sup}$ at some i , then $c_i - 0$ at this i is one of the modes.
If $R_i = f_{sup}$ at some i , then $c_i + 0$ at this i is one of the modes.
If $(L_i + R_i)/2 = f_{sup}$ at some i , then quantiset

$$\{_{1/2}(c_i - 0),\ _{1/2}(c_i + 0)\}^\circ$$

at this i is one of the modes.

Nota bene: The set of all the modes contains the corresponding separate points c_i , as well as one-sided limits $c_i - 0$ and $c_i + 0$, and includes open intervals

$$c_i < x < c_{i+1}\ (i = 1, 2, \ldots , n - 1)$$

for which

$$R_i = L_{i+1} = f_{sup}\ .$$

2.8. Variance

Use the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

In our case we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} (x - \mu)^2 f(x)dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [R_i(x^2 - 2\mu x + \mu^2)(c_{i+1} - x) + L_{i+1}(x^2 - 2\mu x + \mu^2)(x - c_i)]/(c_{i+1} - c_i)\ dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} \{R_i[- x^3 + (2\mu + c_{i+1})x^2 - (\mu^2 + 2\mu c_{i+1})x + \mu^2 c_{i+1}]$$
$$+ L_{i+1}[x^3 - (2\mu + c_i)x^2 + (\mu^2 + 2\mu c_i)x - \mu^2 c_i]\}/(c_{i+1} - c_i)\ dx$$
$$= \Sigma_{i=0}^n \{R_i[- (c_{i+1}^4 - c_i^4)/4 + (2\mu + c_{i+1})(c_{i+1}^3 - c_i^3)/3 - (\mu^2 + 2\mu c_{i+1})(c_{i+1}^2 - c_i^2)/2 + \mu^2 c_{i+1}(c_{i+1} - c_i)]$$
$$+ L_{i+1}[(c_{i+1}^4 - c_i^4)/4 - (2\mu + c_i)(c_{i+1}^3 - c_i^3)/3 + (\mu^2 + 2\mu c_i)(c_{i+1}^2 - c_i^2)/2 - \mu^2 c_i(c_{i+1} - c_i)]\}/(c_{i+1} - c_i)$$
$$= 1/12\ \Sigma_{i=0}^n \{R_i[- 3c_{i+1}^3 - 3c_{i+1}^2 c_i - 3c_{i+1}c_i^2 - 3c_i^3 + (4c_{i+1} + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) - (12\mu c_{i+1} + 6\mu^2)(c_{i+1} + c_i) + 12\mu^2 c_{i+1}]$$
$$+ L_{i+1}[3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - (4c_i + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) + (12\mu c_i + 6\mu^2)(c_{i+1} + c_i) - 12\mu^2 c_i]\}$$
$$= 1/12\ \Sigma_{i=0}^n [R_i(- 3c_{i+1}^3 - 3c_{i+1}^2 c_i - 3c_{i+1}c_i^2 - 3c_i^3 + 4c_{i+1}^3 + 4c_{i+1}^2 c_i + 4c_{i+1}c_i^2 + 8\mu c_{i+1}^2 + 8\mu c_{i+1}c_i + 8\mu c_i^2 - 12\mu c_{i+1}^2 - 12\mu c_{i+1}c_i - 6\mu^2 c_{i+1} - 6\mu^2 c_i + 12\mu^2 c_{i+1})$$
$$+ L_{i+1}(3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - 4c_{i+1}^2 c_i - 4c_{i+1}c_i^2 - 4c_i^3 - 8\mu c_{i+1}^2 - 8\mu c_{i+1}c_i - 8\mu c_i^2 + 12\mu c_{i+1}c_i + 12\mu c_i^2 + 6\mu^2 c_{i+1} + 6\mu^2 c_i - 12\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^n [R_i(c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 - 3c_i^3 - 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i + 8\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)$$
$$+ L_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1}^2 + c_{i+1}c_i + c_i^2 + c_{i+1}c_i + c_i^2 + c_i^2 - 4\mu c_{i+1} - 4\mu c_i - 4\mu c_i + 6\mu^2)$$
$$+ L_{i+1}(c_{i+1}^2 + c_{i+1}c_i + c_i^2 + c_{i+1}c_i + c_{i+1}^2 + c_{i+1}^2 - 4\mu c_{i+1} - 4\mu c_{i+1} - 4\mu c_i + 6\mu^2)]$$
$$= 1/12\ \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2)$$
$$+ L_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)]$$

and, finally,

$$\sigma^2 = \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2)$$
$$+ L_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)]/12$$

where

$$\mu = \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(2c_i + c_{i+1}) + L_{i+1}(c_i + 2c_{i+1})]/6.$$

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

3. Piecewise Linear Probability Density

3.1. Main Definitions

Consider a general one-dimensional bounded-support finite-piecewise linear probability density (Fig. 2).

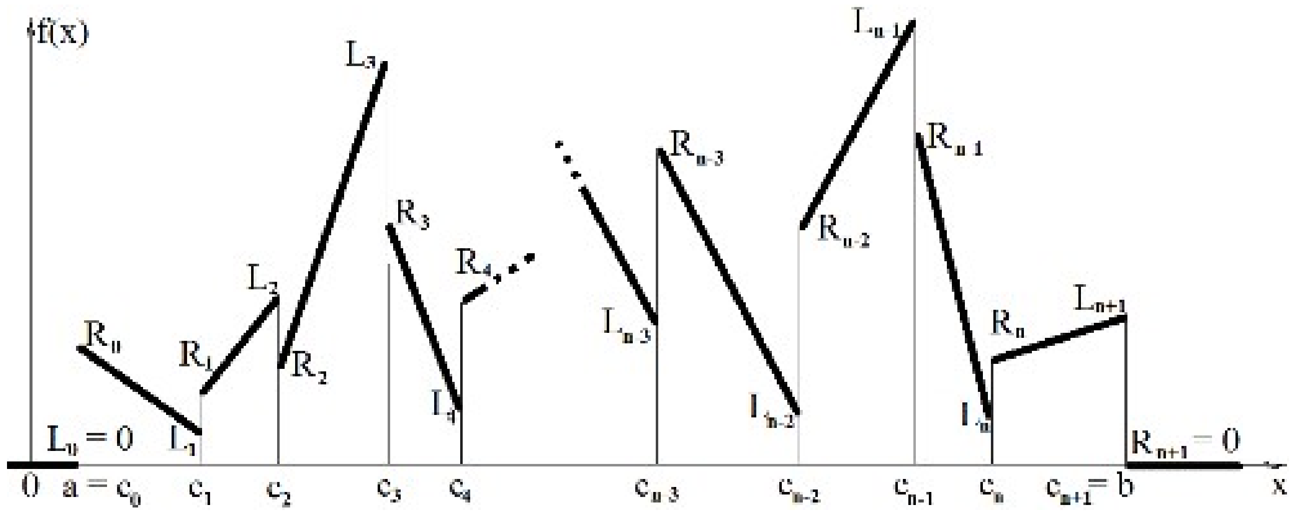

Figure 3. General one-dimensional bounded-support finite-piecewise linear probability density

Here probability density function f(x) is as always non-negative everywhere ($-\infty < x < +\infty$) and can be positive on some so-called support which is a finite segment (closed interval)

$$-\infty < a \le x \le b < +\infty \ (a < b)$$

only. Let n ($n \in N = \{1, 2, ...\}$) intermediate points c_1 , c_2 , c_3 , c_4 , … , c_{n-3} , c_{n-2} , c_{n-1} , c_n in the non-decreasing order so that

$$a \le c_1 \le c_2 \le c_3 \le c_4 \le \ldots \le c_{n-3} \le c_{n-2} \le c_{n-1} \le c_n \le b$$

divide this segment into n + 1 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a ,$$
$$c_{n+1} = b ,$$
$$c(i) = c_i \ (i = 0, 1, 2, \ldots , n + 1).$$

On each of n + 1 open intervals

$$c_i < x < c_{i+1} \ (i = 0, 1, 2, \ldots , n),$$

probability density function f(x) is linear. At n + 2 points

$$c_i \ (i = 0, 1, 2, \ldots , n + 1),$$

f(x) may take any finite non-negative values. The following considerations (possibly excepting mode values below) do not depend on these values. At each of n + 2 points

$$c_i \ (i = 0, 1, 2, \ldots , n + 1),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are any generally different finite non-negative values. Naturally, we have

$$L_0 = 0,$$

$$R_{n+1} = 0.$$

Then on each of n + 1 open intervals

$$c_i < x < c_{i+1} \ (i = 0, 1, 2, \dots , n),$$

linear probability density function

$$f(x) = R_i + (L_{i+1} - R_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= [R_i(c_{i+1} - x) + L_{i+1}(x - c_i)]/(c_{i+1} - c_i).$$

Using short (reduced) notation [Gelimson 2012a], represent non-negative-valued function f(x) via

$$f_{Ra\subseteq[0 , \infty)}(x_{(-\infty , \infty)}) = 0_{(-\infty , c(0)) \cup (c(n+1), \infty)} \cup \cup_{i = 0}^{n} [R_i(c_{i+1} - x) + L_{i+1}(x - c_i)]/(c_{i+1} - c_i)_{(c(i), c(i+1))} \cup \cup_{i = 0}^{n+1} f_{\{c(i)\}}(c_i),$$

or, simplifying $f_{\{c(i)\}}(c_i)$ via identifying [Gelimson 2003a, 2003b] one-point set $\{c(i)\} = \{c_i\}$ at least here with this point $c(i) = c_i$ itself, via

$$f_{Ra\subseteq[0 , \infty)}(x_{(-\infty , \infty)}) = 0_{(-\infty , c(0)) \cup (c(n+1), \infty)} \cup \cup_{i = 0}^{n} [R_i(c_{i+1} - x) + L_{i+1}(x - c_i)]/(c_{i+1} - c_i)_{(c(i), c(i+1))} \cup \cup_{i = 0}^{n+1} f_{c(i)}(c_i),$$

or, further simplifying $f_{c(i)}(c_i)$ via omitting index $c(i) = c_i$ coinciding with argument c_i , which is admissible if and only if argument c_i is explicitly indicated, e.g. here (but NOT after replacing expression $f(c_i)$ via its value, e.g. a number), via

$$f_{Ra\subseteq[0 , \infty)}(x_{(-\infty , \infty)}) = 0_{(-\infty , c(0)) \cup (c(n+1), \infty)} \cup \cup_{i = 0}^{n} [R_i(c_{i+1} - x) + L_{i+1}(x - c_i)]/(c_{i+1} - c_i)_{(c(i), c(i+1))} \cup \cup_{i = 0}^{n+1} f(c_i)$$

on the whole real axis $(-\infty , \infty)$

where

index $[0 , \infty)$ in $f_{[0 , \infty)}$ indicates the domain of dependent variable f and hence the range of function f(x),

index $(-\infty , \infty)$ in $x_{(-\infty , \infty)}$ indicates the range of x and hence the domain of one-argument function f(x),

index $(-\infty , c(0)) \cup (c(n+1), \infty) = (-\infty , c_0) \cup (c_{n+1} , \infty)$ in $0_{(-\infty , c(0))}$ indicates that function f(x) = 0 on its subdomain $(-\infty , c(0)) \cup (c(n+1), \infty) = (-\infty , c_0) \cup (c_{n+1} , \infty)$,

symbol $\cup$ unifies subfunctions on subdomains similarly to symbol $\cup$ in set theory and can be also indexed with an index range,

bounds 0 and n of index i in $\cup_{i = 0}^{n}$ indicate that the range of index i is $\{0, 1, 2, \dots , n\}$,

index $(c(i), c(i+1)) = (c_i , c_{i+1})$ in $\{[R_i(c_{i+1} - x) + L_{i+1}(x - c_i)]/(c_{i+1} - c_i)\}_{(c(i), c(i+1))}$ indicates that function $f(x) = [R_i(c_{i+1} - x)_{+L_{i+1}}(x - c_i)]/(c_{i+1} - c_i)$ on its subdomain $(c(i), c(i+1)) = (c_i , c_{i+1})$,

index $\{c(i)\} = \{c_i\}$ in $f_{\{c(i)\}}(c_i)$ indicates that function $f(x) = f(c_i)$ on its subdomain $\{c(i)\} = \{c_i\}$.

3.2. Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

In our case we have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} f(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [R_i(c_{i+1} - x) + L_{i+1}(x - c_i)]/(c_{i+1} - c_i)\, dx$$
$$= \Sigma_{i=0}^n \{R_i[c_{i+1}(c_{i+1} - c_i) - (c_{i+1}^2 - c_i^2)/2] + L_{i+1}[(c_{i+1}^2 - c_i^2)/2 - c_i(c_{i+1} - c_i)]\}/(c_{i+1} - c_i)$$
$$= \Sigma_{i=0}^n \{R_i[c_{i+1} - (c_{i+1} + c_i)/2] + L_{i+1}[(c_{i+1} + c_i)/2 - c_i]\}$$
$$= \Sigma_{i=0}^n (R_i + L_{i+1})(c_{i+1} - c_i)/2.$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the n + 1 rectangular trapezoids.

Therefore, to provide a possible (an admissible) probability density function, necessary and sufficient integral normalization condition

$$\Sigma_{i=0}^n (R_i + L_{i+1})(c_{i+1} - c_i) = 2$$

has to be satisfied.

3.3. Normalization Algorithm

Nota bene: The obtained normalization condition is one condition only for

$$(n + 1) + (n + 1) + (n + 2) = 3n + 4$$

unknowns

$$R_i \ (i = 0, 1, 2, \dots , n),$$
$$L_i \ (i = 1, 2, 3, \dots , n + 1),$$
$$c_i \ (i = 0, 1, 2, \dots , n + 1).$$

Additionally,

$$R_i \geq 0 \ (i = 0, 1, 2, \dots , n),$$
$$L_i \geq 0 \ (i = 1, 2, 3, \dots , n + 1),$$
$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \dots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} .$$

Generally, it is not possible to simply take any admissible values of

$$3n + 4 - 1 = 3n + 3$$

unknowns and then to determine the value of the remaining unknown via this condition because it can happen that this value is inadmissible.

A natural idea, way, and algorithm to avoid this difficulty are as follows:

1. Fix

$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \dots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} .$$

2. Take any

$$R_i' \geq 0 \ (i = 0, 1, 2, \dots , n),$$
$$L_i' \geq 0 \ (i = 1, 2, 3, \dots , n + 1)$$

so that there is at least one namely positive number among these 2n + 2 non-negative numbers.

3. Let

$$R_i \ (i = 0, 1, 2, \dots , n),$$
$$L_i \ (i = 1, 2, 3, \dots , n + 1)$$

be proportional to

$$R_i' \geq 0 \ (i = 0, 1, 2, \dots , n),$$
$$L_i' \geq 0 \ (i = 1, 2, 3, \dots , n + 1),$$

respectively, with a common namely positive factor k so that

$$R_i = kR_i' \ (i = 0, 1, 2, \dots , n),$$
$$L_i = kL_i' \ (i = 1, 2, 3, \dots , n + 1).$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$\Sigma_{i=0}^n (R_i + L_{i+1})(c_{i+1} - c_i) = 2$$

so that

$$k = 2 / \Sigma_{i=0}^n (R_i' + L_{i+1}')(c_{i+1} - c_i).$$

5. Explicitly determine

$$R_i = kR_i' \ (i = 0, 1, 2, \dots , n),$$
$$L_i = kL_i' \ (i = 1, 2, 3, \dots , n + 1).$$

3.4. Integral (Cumulative) Probability Distribution Function

Integral (cumulative) probability distribution function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

is probability $P(X \leq x)$ that real-number random variable X takes a real-number value not greater than x .

For $x \leq a = c_0$, this definition gives

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = 0.$$

For $x \geq b = c_{n+1}$, this definition gives

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = 1.$$

For $c_0 = a < x < b = c_{n+1}$, we can use the following natural idea, way, and algorithm:

1. Determine such value j of index i that

$$c_j \leq x < c_{j+1} .$$

There exists such value j and namely the only. Indeed, consider set

$$\{i \mid i \in \{0, 1, 2, \ldots , n\}, c_j \leq x\}.$$

It is non-empty because it contains at least 0 for which

$$c_0 = a < x$$

and hence

$$c_0 \leq x .$$

It is finite and strictly ordered by relation < . Therefore, there exists its maximal element

$$j = \max\{i \mid i \in \{0, 1, 2, \ldots , n\}, c_j \leq x\},$$

and this maximal element is namely the only. Then for this maximal element j , in addition to

$$c_j \leq x ,$$

inequality

$$x < c_{j+1}$$

also holds. Indeed, otherwise, we would have

$$x \geq c_{j+1}$$

and

$$c_{j+1} \leq x ,$$

so that this j could not be namely the maximal element of this set.

2. Determine

$$\begin{gathered}
F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{a}^{x} f(t)dt = \int_{c(0)}^{x} f(t)dt \\
= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} f(t)dt + \int_{c(j)}^{x} f(t)dt \\
= \Sigma_{i=0}^{j-1} \int_{c(i)}^{c(i+1)} [R_i(c_{i+1} - t) + L_{i+1}(t - c_i)]/(c_{i+1} - c_i)\, dt \\
+ \int_{c(j)}^{x} [R_j(c_{j+1} - t) + L_{j+1}(t - c_j)]/(c_{j+1} - c_j)\, dt \\
= \Sigma_{i=0}^{j-1} \{R_i[c_{i+1}(c_{i+1} - c_i) - (c_{i+1}^2 - c_i^2)/2] + L_{i+1}[(c_{i+1}^2 - c_i^2)/2 - c_i(c_{i+1} - c_i)]\}/(c_{i+1} - c_i) \\
+ \{R_j[c_{j+1}(x - c_j) - (x^2 - c_j^2)/2] + L_{j+1}[(x^2 - c_j^2)/2 - c_j(x - c_j)]\}/(c_{j+1} - c_j) \\
= \Sigma_{i=0}^{j-1} \{R_i[c_{i+1} - (c_{i+1} + c_i)/2] + L_{i+1}[(c_{i+1} + c_i)/2 - c_i]\} \\
+ \{R_j[c_{j+1}(x - c_j) - (x^2 - c_j^2)/2] + L_{j+1}[(x^2 - c_j^2)/2 - c_j(x - c_j)]\}/(c_{j+1} - c_j) \\
= \Sigma_{i=0}^{j-1} (R_i + L_{i+1})(c_{i+1} - c_i)/2 + \{R_j[c_{j+1}(x - c_j) - (x^2 - c_j^2)/2] + L_{j+1}[(x^2 - c_j^2)/2 - c_j(x - c_j)]\}/(c_{j+1} - c_j).
\end{gathered}$$

3.5. Mean Value (Mathematical Expectation)

Use the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx\ .$$

In our case we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$

$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} xf(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [R_i(c_{i+1}x - x^2) + L_{i+1}(x^2 - c_ix)]/(c_{i+1} - c_i)\ dx$$

$$= \Sigma_{i=0}^n \{R_i[c_{i+1}(c_{i+1}{}^2 - c_i{}^2)/2 - (c_{i+1}{}^3 - c_i{}^3)/3] + L_{i+1}[(c_{i+1}{}^3 - c_i{}^3)/3 - c_i(c_{i+1}{}^2 - c_i{}^2)/2]/(c_{i+1} - c_i)\}$$

$$= 1/6\ \Sigma_{i=0}^n \{R_i[3c_{i+1}(c_{i+1} + c_i) - 2(c_{i+1}{}^2 + c_{i+1}c_i + c_i{}^2)] + L_{i+1}[2(c_{i+1}{}^2 + c_{i+1}c_i + c_i{}^2) - 3c_i(c_{i+1} + c_i)]\}$$

$$= 1/6\ \Sigma_{i=0}^n [R_i(c_{i+1}{}^2 + c_{i+1}c_i - 2c_i{}^2) + L_{i+1}(2c_{i+1}{}^2 - c_{i+1}c_i - c_i{}^2)]$$

$$= 1/6\ \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1} + 2c_i) + L_{i+1}(2c_{i+1} + c_i)]$$

and, finally,

$$\mu = \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(2c_i + c_{i+1}) + L_{i+1}(c_i + 2c_{i+1})]/6.$$

3.6. Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \le \nu) \ge 1/2$$

and

$$P(X \ge \nu) \ge 1/2.$$

For a continual real-number random variable X ,

$$P(X \le \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \ge \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the following natural idea, way, and algorithm:

1. First consider

$$c_i \ (i = 0, 1, 2, \dots , n + 1)$$

not far from μ and determine both

$$L = \max\{i \mid \int_{-\infty}^{c(i)} f(x)dx < 1/2\}$$

and

$$R = \min\{i \mid \int_{c(i)}^{+\infty} f(x)dx < 1/2\}.$$

Then both

$$\int_{-\infty}^{c(L+1)} f(x)dx \ge 1/2$$

and

$$\int_{c(R-1)}^{+\infty} f(x)dx \ge 1/2.$$

2. On half-closed interval

$$c(L) = c_L < \nu \le c_{L+1} = c(L+1),$$

determine

$$\nu_{min} = \inf\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}.$$

3. On half-closed interval

$$c(R-1) = c_{R-1} \le \nu < c_R = c(R),$$

determine

$$\nu_{max} = \sup\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\}.$$

4. Then the set of all the median values ν is the interval whose endpoints are

$$\nu_{min} \le \nu_{max}$$

each of which is included into the interval if and only if the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and/or

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

respectively.

Notata bene:

1. If

$$\nu_{min} = \nu_{max} ,$$

then the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

hence the closed interval

$$\nu_{min} \le \nu \le \nu_{max}$$

contains the only median value

$$\nu = \nu_{min} = \nu_{max} .$$

2. If

$$\nu_{min} < \nu_{max} ,$$

then the integral of f(x) on the interval whose endpoints are ν_{min} and ν_{max} vanishes independently of their including or excluding. Hence on this interval, non-negative probability density function f(x) also vanishes possibly excepting points whose set has zero measure (in our case, a finite set).

3.7. Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density function f(x) takes its maximum value f_{max} . For a continual probability density function, generalize this definition in the following directions:

1. Replace the maximum value f_{max} with the supremum value f_{sup} which always exists. The reason is that it is possible (for a piecewise linear probability density, too) that function f(x) is discontinuous and does not take the supremum value f_{sup} so that the maximum value f_{max} does not exist at all.
2. Extend the range of function f(x), i.e. the set of values function f(x) really (truly) takes, via all the limiting points of this set. Then the extended range is a closed set and contains, in particular, the supremum value f_{sup} .
3. Extend the domain of function f(x), i.e. the set of points at which function f(x) is properly defined, via all the limiting points of this set. Then the extended domain is a closed set which contains all its limiting points.
4. Admit modes to also correspond to the one-sided limits of function f(x) separately if necessary. This is important for discontinuous function f(x) with jumps.
5. At any interval endpoint c_i , along with the given value of $f(c_i)$, take into account the one-sided limits L_i and R_i of function f(x), e.g. any of the following reasonable options for value $f(c_i)$:

5.1. Take the given value of $f(c_i)$ itself.

5.2. Take

$$f(c_i) = \max\{L_i , R_i\}.$$

5.3. Take

$$f(c_i) = (L_i + R_i)/2.$$

6. At any interval endpoint c_i , along with c_i itself, take into account the one-sided limiting points c_i - 0 and c_i + 0 corresponding to one-sided limits L_i and R_i of function f(x), respectively, e.g. any of the following reasonable options for c_i :

6.1. Take the given value of c_i itself.

6.2. For modes, rather than c_i , consider

$$c_i - 0 \text{ if } L_i > R_i ,$$
$$c_i + 0 \text{ if } L_i < R_i ,$$

and quantiset [Gelimson 2003a, 2003b]

$$\{{}_{1/2}(c_i - 0), {}_{1/2}(c_i + 0)\}° \text{ if } L_i = R_i .$$

This quantiset consists of two quantielements

$${}_{1/2}(c_i - 0), {}_{1/2}(c_i + 0)$$

with bases

$$c_i - 0, c_i + 0,$$

respectively.

Here each of elements c_i - 0 and c_i + 0 has quantity 1/2 so that the total unit quantity is equally divided between these both elements.

In particular, for a piecewise linear probability density function f(x), anyone of the following values can reasonably play the role of f_{sup} :

$$\max\{\max\{f(c_i) \mid i = 0, 1, 2, \dots , n + 1\}, \max\{L_i \mid i = 0, 1, \dots , n + 1\}, \max\{R_i \mid i = 0, 1, \dots , n + 1\}\},$$
$$\max\{\max\{f(c_i) \mid i = 0, 1, 2, \dots , n + 1\}, \max\{(L_i + R_i)/2 \mid i = 0, 1, 2, \dots , n + 1\}\},$$
$$\max\{\max\{L_i \mid i = 0, 1, 2, \dots , n + 1\}, \max\{R_i \mid i = 0, 1, 2, \dots , n + 1\}\},$$
$$\max\{(L_i + R_i)/2 \mid i = 0, 1, 2, \dots , n + 1\}.$$

If $f(c_i) = f_{sup}$ at some i , then c_i at this i is one of the modes.

If $L_i = f_{sup}$ at some i , then c_i - 0 at this i is one of the modes.

If $R_i = f_{sup}$ at some i , then c_i + 0 at this i is one of the modes.

If $(L_i + R_i)/2 = f_{sup}$ at some i , then quantiset

$$\{_{1/2}(c_i - 0),\ _{1/2}(c_i + 0)\}^\circ$$

at this i is one of the modes.

Nota bene: The set of all the modes contains the corresponding separate points c_i , as well as one-sided limits $c_i - 0$ and $c_i + 0$, and includes open intervals

$$c_i < x < c_{i+1}\ (i = 1, 2, \dots , n - 1)$$

for which

$$R_i = L_{i+1} = f_{sup}\ .$$

3.8. Variance

Use the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

In our case we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} (x - \mu)^2 f(x)dx$$

$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [R_i(x^2 - 2\mu x + \mu^2)(c_{i+1} - x) + L_{i+1}(x^2 - 2\mu x + \mu^2)(x - c_i)]/(c_{i+1} - c_i)\, dx$$

$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} \{R_i[- x^3 + (2\mu + c_{i+1})x^2 - (\mu^2 + 2\mu c_{i+1})x + \mu^2 c_{i+1}]$$

$$+ L_{i+1}[x^3 - (2\mu + c_i)x^2 + (\mu^2 + 2\mu c_i)x - \mu^2 c_i]\}/(c_{i+1} - c_i)\, dx$$

$$= \Sigma_{i=0}^n \{R_i[- (c_{i+1}^4 - c_i^4)/4 + (2\mu + c_{i+1})(c_{i+1}^3 - c_i^3)/3 - (\mu^2 + 2\mu c_{i+1})(c_{i+1}^2 - c_i^2)/2 + \mu^2 c_{i+1}(c_{i+1} - c_i)]$$

$$+ L_{i+1}[(c_{i+1}^4 - c_i^4)/4 - (2\mu + c_i)(c_{i+1}^3 - c_i^3)/3 + (\mu^2 + 2\mu c_i)(c_{i+1}^2 - c_i^2)/2 - \mu^2 c_i(c_{i+1} - c_i)]\}/(c_{i+1} - c_i)$$

$$= 1/12\ \Sigma_{i=0}^n \{R_i[- 3c_{i+1}^3 - 3c_{i+1}^2 c_i - 3c_{i+1}c_i^2 - 3c_i^3 + (4c_{i+1} + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) - (12\mu c_{i+1} + 6\mu^2)(c_{i+1} + c_i) + 12\mu^2 c_{i+1}]$$

$$+ L_{i+1}[3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - (4c_i + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) + (12\mu c_i + 6\mu^2)(c_{i+1} + c_i) - 12\mu^2 c_i]\}$$

$$= 1/12\ \Sigma_{i=0}^n [R_i(- 3c_{i+1}^3 - 3c_{i+1}^2 c_i - 3c_{i+1}c_i^2 - 3c_i^3 + 4c_{i+1}^3 + 4c_{i+1}^2 c_i + 4c_{i+1}c_i^2 + 8\mu c_{i+1}^2 + 8\mu c_{i+1}c_i + 8\mu c_i^2 - 12\mu c_{i+1}^2 - 12\mu c_{i+1}c_i - 6\mu^2 c_{i+1} - 6\mu^2 c_i + 12\mu^2 c_{i+1})$$

$$+ L_{i+1}(3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - 4c_{i+1}^2 c_i - 4c_{i+1}c_i^2 - 4c_i^3 - 8\mu c_{i+1}^2 - 8\mu c_{i+1}c_i - 8\mu c_i^2 + 12\mu c_{i+1}c_i + 12\mu c_i^2 + 6\mu^2 c_{i+1} + 6\mu^2 c_i - 12\mu^2 c_i)]$$

$$= 1/12\ \Sigma_{i=0}^n [R_i(c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 - 3c_i^3 - 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i + 8\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)$$

$$+ L_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$

$$= 1/12\ \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1}^2 + c_{i+1}c_i + c_i^2 + c_{i+1}c_i + c_i^2 + c_i^2 - 4\mu c_{i+1} - 4\mu c_i - 4\mu c_i + 6\mu^2)$$

$$+ L_{i+1}(c_{i+1}^2 + c_{i+1}c_i + c_i^2 + c_{i+1}c_i + c_{i+1}^2 + c_{i+1}^2 - 4\mu c_{i+1} - 4\mu c_{i+1} - 4\mu c_i + 6\mu^2)]$$

$$= 1/12\ \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2)$$

$$+ L_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)]$$

and, finally,

$$\sigma^2 = \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2)$$

$$+ L_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)]/12$$

where

$$\mu = \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(2c_i + c_{i+1}) + L_{i+1}(c_i + 2c_{i+1})]/6.$$

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

3. General Polygonal, or Piecewise Linear Continuous, Probability Density

3.1. Main Definitions

Consider a general one-dimensional polygonal, or piecewise linear continuous, probability density (Fig. 2) as a particular case of a general one-dimensional piecewise linear probability density.

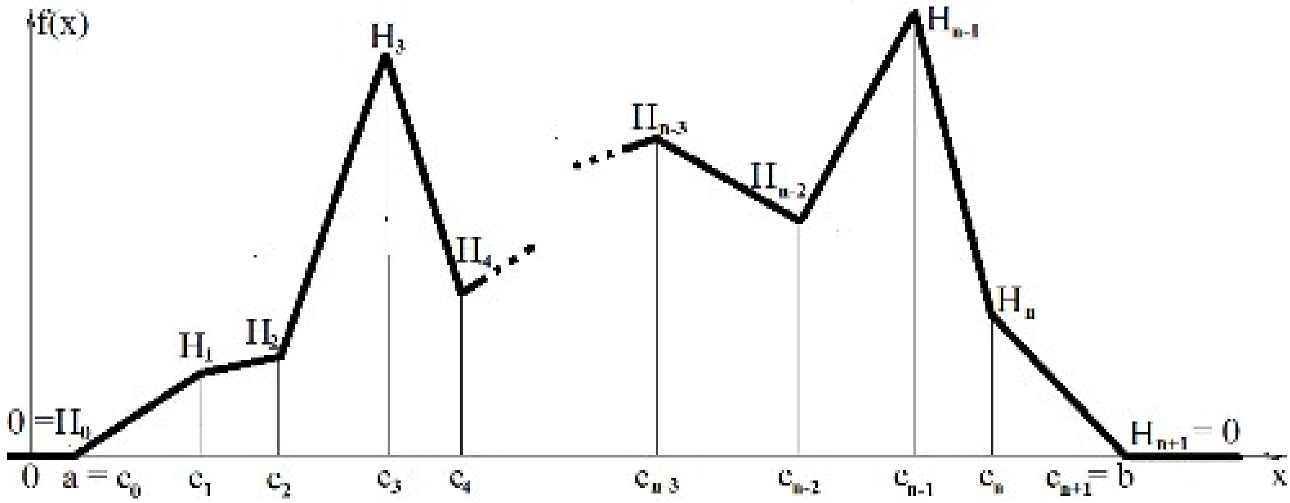

Figure 4. General one-dimensional bounded-support finite-piecewise linear continuous probability density

Here probability density function $f(x)$ is as always non-negative everywhere $(-\infty < x < +\infty)$ and can be positive on some finite segment (closed interval)

$$-\infty < a \le x \le b < +\infty \ (a < b)$$

only. Let n ($n \in N = \{1, 2, ...\}$) intermediate points $c_1, c_2, c_3, c_4, \dots, c_{n-3}, c_{n-2}, c_{n-1}, c_n$ in the non-decreasing order so that

$$a \le c_1 \le c_2 \le c_3 \le c_4 \le \dots \le c_{n-3} \le c_{n-2} \le c_{n-1} \le c_n \le b$$

divide this segment into $n + 1$ parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a,$$

$$c_{n+1} = b,$$

$$c(i) = c_i \ (i = 0, 1, 2, \dots, n + 1).$$

On each of $n + 1$ closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1, 2, \dots, n),$$

probability density function $f(x)$ is linear. At $n + 2$ points

$$c_i \ (i = 0, 1, 2, \dots, n + 1),$$

$f(x)$ takes finite non-negative values

$$H_i = f(c_i),$$

respectively. Naturally, we have

$$H_0 = 0,$$

$$H_{n+1} = 0.$$

Note that

$$H_i = f(c_i)\ (i = 1, 2, \dots , n)$$

may be any finite non-negative values. At each of n + 2 points

$$c_i\ (i = 0, 1, 2, \dots , n + 1),$$

left and right one-sided limits

$$\lim f(x) = L_i\ (x \to c_i - 0),$$
$$\lim f(x) = R_i\ (x \to c_i + 0)$$

are equal to one another and coincide with $f(c_i)$. Therefore, we obtain

$$H_i = L_i = R_i\ (i = 0, 1, 2, \dots , n + 1),$$

which makes it possible to apply the above formulas for a piecewise linear probability density to a piecewise linear continuous probability density.

Then on each of n + 1 closed intervals

$$c_i \le x \le c_{i+1}\ (i = 0, 1, 2, \dots , n),$$

linear probability density function

$$f(x) = H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= [H_i(c_{i+1} - x) + H_{i+1}(x - c_i)]/(c_{i+1} - c_i).$$

Use short (reduced) notation [Gelimson 2012a] and the corresponding formula for a piecewise linear probability density. Then in our continuous case we represent non-negative-valued function f(x) via

$$f_{[0 , \infty)}(x_{(-\infty , \infty)}) = 0_{(-\infty , c(0)) \cup [c(n+1), \infty)} \cup \cup_{i = 0}{}^{n} [H_i(c_{i+1} - x) + H_{i+1}(x - c_i)]/(c_{i+1} - c_i)_{[c(i), c(i+1))}$$

on the whole real axis $(-\infty , \infty)$.

Integral (cumulative) probability distribution function

$$F(x) = P(X \le x) = \int_{-\infty}{}^{x} f(t)dt$$

is probability $P(X \le x)$ that real-number random variable X takes a real-number value not greater than x .

3.2. Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

Use the corresponding formula for a piecewise linear probability density. Then in our continuous case we determine

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$
$$= \Sigma_{i=0}^n (R_i + L_{i+1})(c_{i+1} - c_i)/2$$
$$= \Sigma_{i=0}^n (H_i + H_{i+1})(c_{i+1} - c_i)/2$$
$$= \Sigma_{i=0}^n H_i(c_{i+1} - c_i)/2 + \Sigma_{i=0}^n H_{i+1}(c_{i+1} - c_i)/2.$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the n + 1 rectangular trapezoids, among them 2 rectangular triangles at the endpoints a and b .

Now use

$$H_0 = 0,$$
$$H_{n+1} = 0.$$

Then

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \Sigma_{i=1}^n H_i(c_{i+1} - c_i)/2 + \Sigma_{i=1}^n H_i(c_i - c_{i-1})/2$$
$$= \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})/2.$$

Therefore, to provide a possible (an admissible) probability density function, necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1}) = 2$$

has to be satisfied.

3.3. Normalization Algorithm

Nota bene: The obtained normalization condition is one condition only for

$$n + (n + 2) = 2n + 2$$

unknowns

$$H_i \ (i = 1, 2, \ldots , n),$$
$$c_i \ (i = 0, 1, 2, \ldots , n + 1).$$

Additionally,

$$H_i \geq 0 \ (i = 1, 2, 3, \ldots , n),$$
$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \ldots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} .$$

Generally, it is not possible to simply take any admissible values of

$$2n + 2 - 1 = 2n + 1$$

unknowns and then to determine the value of the remaining unknown via this condition because it can happen that this value is inadmissible.

A natural idea, way, and algorithm to avoid this difficulty are as follows:

1. Fix

$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \ldots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} .$$

2. Take any

$$H_i' \geq 0 \ (i = 1, 2, \ldots , n)$$

so that there is at least one namely positive number among these n non-negative numbers.

3. Let

$$H_i \ (i = 1, 2, 3, \ldots , n)$$

be proportional to

$$H_i' \geq 0 \ (i = 1, 2, 3, \ldots , n),$$

respectively, with a common namely positive factor k so that

$$H_i = kH_i' \ (i = 1, 2, 3, \ldots , n).$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1}) = 2$$

so that

$$k = 2 / \Sigma_{i=0}^{n} H_i'(c_{i+1} - c_{i-1}).$$

5. Explicitly determine

$$H_i = kH_i' \ (i = 1, 2, 3, \ldots , n).$$

3.4. Mean Value (Mathematical Expectation)

Take the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx \,.$$

Use the corresponding formula for a piecewise linear probability density. Then in our continuous case we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(2c_i + c_{i+1}) + L_{i+1}(c_i + 2c_{i+1})]/6$$
$$= 1/6\ \Sigma_{i=0}^n (c_{i+1} - c_i)[H_i(c_{i+1} + 2c_i) + H_{i+1}(2c_{i+1} + c_i)]$$
$$= 1/6\ \Sigma_{i=0}^n (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6\ \Sigma_{i=0}^n (c_{i+1} - c_i)H_{i+1}(2c_{i+1} + c_i)$$
$$= 1/6\ \Sigma_{i=1}^n (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6\ \Sigma_{i=1}^n (c_i - c_{i-1})H_i(2c_i + c_{i-1})$$
$$= 1/6\ \Sigma_{i=1}^n H_i[(c_{i+1} - c_i)(c_{i+1} + 2c_i) + (c_i - c_{i-1})(2c_i + c_{i-1})]$$
$$= 1/6\ \Sigma_{i=1}^n H_i(c_{i+1}^2 + c_{i+1}c_i - 2c_i^2 + 2c_i^2 - c_ic_{i-1} - c_{i-1}^2)$$
$$= 1/6\ \Sigma_{i=1}^n H_i(c_{i+1}^2 + c_{i+1}c_i - c_ic_{i-1} - c_{i-1}^2)$$
$$= 1/6\ \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})$$

and, finally,

$$\mu = \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

3.5. Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \leq \nu) \geq 1/2$$

and

$$P(X \geq \nu) \geq 1/2.$$

For a continual real-number random variable X ,

$$P(X \leq \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \geq \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the same natural idea, way, and algorithm as for a general one-dimensional piecewise linear probability density but, naturally, with the formulas for a general one-dimensional piecewise linear continuous probability density.

3.6. Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density function $f(x)$ takes its maximum value f_{max} .

In particular, for a piecewise linear continuous probability density function $f(x)$,

$$f_{max} = \max\{f(c_i) \mid i = 1, 2, \dots , n\}.$$

If $f(x) = f_{max}$ at some x , then this x is one of the modes.

In particular, if $f(c_i) = f_{max}$ at some i , then c_i at this i is one of the modes.

Nota bene: The set of all the modes both contains separate points

$$c_i \ (i = 1, 2, \dots , n)$$

for which

$$f(c_i) = f_{sup} = f_{max}$$

and includes closed intervals

$$c_i \le x \le c_{i+1} \ (i = 1, 2, \dots , n - 1)$$

for which

$$f(c_i) = f(c_{i+1}) = f_{sup} = f_{max} .$$

3.7. Variance

Take the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx .$$

Use the corresponding formula for a piecewise linear probability density. Then in our continuous case we determine

$$\begin{aligned}
\sigma^2 &= \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2) \\
&\quad + L_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)]/12 \\
&= \Sigma_{i=0}^n (c_{i+1} - c_i)[H_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2) \\
&\quad + H_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)]/12 \\
&= \Sigma_{i=0}^n (c_{i+1} - c_i)H_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2)/12 \\
&\quad + \Sigma_{i=0}^n (c_{i+1} - c_i)H_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)/12 \\
&= \Sigma_{i=1}^n (c_{i+1} - c_i)H_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2)/12 \\
&\quad + \Sigma_{i=0}^{n-1} (c_{i+1} - c_i)H_{i+1}(3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 8\mu c_{i+1} - 4\mu c_i + 6\mu^2)/12 \\
&= \Sigma_{i=1}^n (c_{i+1} - c_i)H_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 - 4\mu c_{i+1} - 8\mu c_i + 6\mu^2)/12 \\
&\quad + \Sigma_{i=1}^n (c_i - c_{i-1})H_i(3c_i^2 + 2c_ic_{i-1} + c_{i-1}^2 - 8\mu c_i - 4\mu c_{i-1} + 6\mu^2)/12 \\
&= \Sigma_{i=1}^n H_i(c_{i+1}^3 + 2c_{i+1}^2 c_i + 3c_{i+1}c_i^2 - 4\mu c_{i+1}^2 - 8\mu c_{i+1}c_i + 6\mu^2 c_{i+1} \\
&\quad - c_{i+1}^2c_i - 2c_{i+1}c_i^2 - 3c_i^3 + 4\mu c_{i+1}c_i + 8\mu c_i^2 - 6\mu^2 c_i \\
&\quad + 3c_i^3 + 2c_i^2c_{i-1} + c_ic_{i-1}^2 - 8\mu c_i^2 - 4\mu c_ic_{i-1} + 6\mu^2 c_i \\
&\quad - 3c_i^2c_{i-1} - 2c_ic_{i-1}^2 - c_{i-1}^3 + 8\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 6\mu^2 c_{i-1})/12 \\
&= \Sigma_{i=1}^n H_i(c_{i+1}^3 + c_{i+1}^2c_i + c_{i+1}c_i^2 - c_i^2c_{i-1} - c_ic_{i-1}^2 - c_{i-1}^3 \\
&\quad - 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i + 4\mu c_ic_{i-1} + 4\mu c_{i-1}^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_{i-1})/12 \\
&= \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_{i-1} + c_{i-1}^2 + c_{i+1}c_i + c_ic_{i-1} + c_i^2 \\
&\quad - 4\mu c_{i+1} - 4\mu c_{i-1} - 4\mu c_i + 6\mu^2)/12
\end{aligned}$$

and, finally,

$$\sigma^2 = \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})[c_{i+1}^2 + c_i^2 + c_{i-1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_ic_{i-1} - 4\mu(c_{i+1} + c_i + c_{i-1}) + 6\mu^2]/12$$

where

$$\mu = \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

4. Tetragonal Probability Density

4.1. Main Definitions

A tetragonal probability density (Fig. 3) is a particular case of a general one-dimensional piecewise linear continuous probability density for n = 2 and further of a general one-dimensional piecewise linear probability density. Therefore, directly apply the above formulas for a general one-dimensional piecewise linear continuous probability density to a tetragonal probability density.

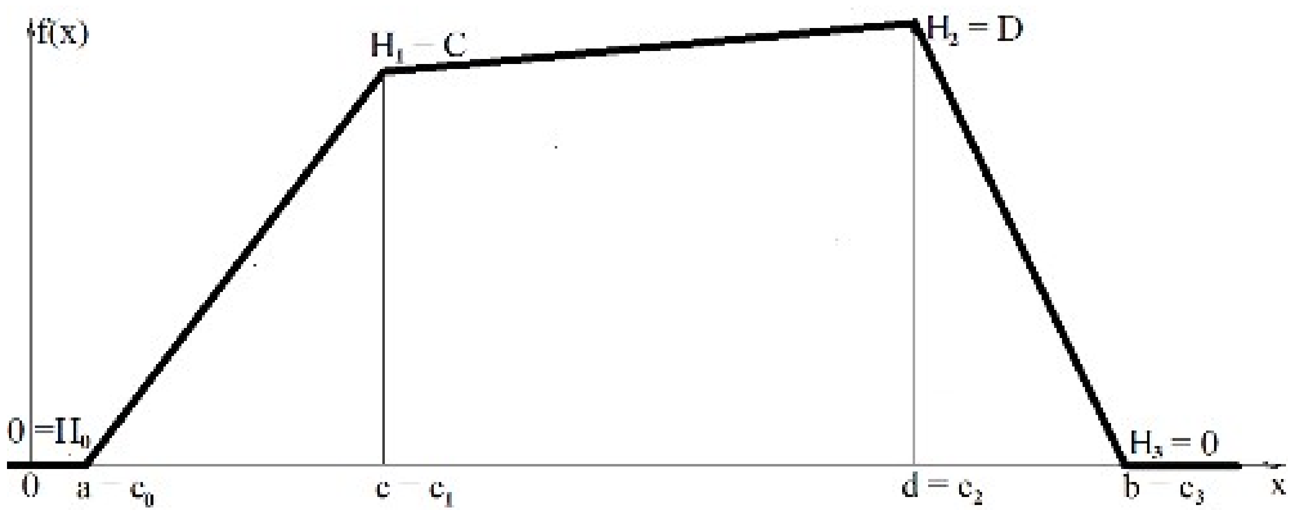

Figure 5. Tetragonal probability density

Here probability density function f(x) is as always non-negative everywhere ($-\infty < x < +\infty$) and can be positive on some finite segment (closed interval)

$$-\infty < a \le x \le b < +\infty \ (a < b)$$

only. Let n = 2 intermediate points $c = c_1$ and $d = c_2$ in the non-decreasing order so that

$$a \le c_1 \le c_2 \le b$$

divide this segment into n + 1 = 3 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a \ ,$$
$$c_3 = b \ ,$$
$$c(i) = c_i \ (i = 0, 1, 2, 3).$$

On each of n + 1 = 3 closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1, 2),$$

probability density function f(x) is linear. At n + 2 = 4 points

$$c_i \ (i = 0, 1, 2, 3),$$

f(x) takes finite non-negative values

$$H_i = f(c_i),$$

respectively. Naturally, we have

$$H_0 = 0,$$
$$H_3 = 0.$$

Note that

$$H_i = f(c_i) \ (i = 1, 2)$$

with additional natural notation

$$C = H_1 ,$$
$$D = H_2$$

for values f(x) at points

$$c = c_1 ,$$
$$d = c_2 ,$$

respectively, may be any finite non-negative values. At each of n + 2 = 4 points

$$c_i \ (i = 0, 1, 2, 3),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are equal to one another and coincide with $f(c_i)$. Therefore, we obtain

$$H_i = L_i = R_i \ (i = 0, 1, 2, 3).$$

Then on each of n + 1 = 3 closed intervals

$$c_i \leq x \leq c_{i+1} \ (i = 0, 1, 2),$$

linear probability density function

$$f(x) = H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= [H_i(c_{i+1} - x) + H_{i+1}(x - c_i)]/(c_{i+1} - c_i).$$

Use short (reduced) notation [Gelimson 2012a] and the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 2 we represent non-negative-valued function f(x) via

$$f_{[0\,,\,\infty)}(x_{(-\infty\,,\,\infty)}) = 0_{(-\infty\,,\,c(0))\,\cup\,[c(3),\,\infty)} \cup \cup_{i=0}{}^{2}\, [H_i(c_{i+1} - x) + H_{i+1}(x - c_i)]/(c_{i+1} - c_i)_{[c(i),\,c(i+1))}$$

on the whole real axis (-∞ , ∞).
Using

$$c_0 = a ,$$
$$c_1 = c ,$$
$$c_2 = d ,$$
$$c_3 = b ,$$
$$H_0 = 0,$$
$$H_1 = C ,$$
$$H_2 = D ,$$
$$H_3 = 0,$$

we obtain

$$f_{[0\,,\,\infty)}(x_{(-\infty\,,\,\infty)}) = 0_{(-\infty\,,\,c(0))\,\cup\,[c(3),\,\infty)} \cup [H_0(c_1 - x) + H_1(x - c_0)]/(c_1 - c_0)\}_{[c(0),\,c(1))}$$
$$\cup\ [H_1(c_2 - x) + H_2(x - c_1)]/(c_2 - c_1)\}_{[c(1),\,c(2))} \cup [H_2(c_3 - x) + H_3(x - c_2)]/(c_3 - c_2)\}_{[c(2),\,c(3))} ,$$
$$f_{[0\,,\,\infty)}(x_{(-\infty,\,\infty)}) = 0_{(-\infty,\,a)\cup[b\,,\,\infty)} \cup C(x - a)/(c - a)_{[a\,,\,c)} \cup [C(d - x)+D(x - c)]/(d - c)_{[c\,,\,d)} \cup D(b - x)/(b - d)_{[d\,,\,b)}.$$

Integral (cumulative) probability distribution function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

is probability $P(X \leq x)$ that real-number random variable X takes a real-number value not greater than x .

4.2. Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

Use the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 2 we determine

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$
$$= \Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1})/2$$
$$= \Sigma_{i=1}^{2} H_i(c_{i+1} - c_{i-1})/2.$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the n + 1 = 3 rectangular trapezoids, among them 2 rectangular triangles at the endpoints a and b .

Therefore, to provide a possible (an admissible) probability density function, necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^{2} H_i(c_{i+1} - c_{i-1}) = 2$$

has to be satisfied.

Using

$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = d\ ,$$
$$c_3 = b\ ,$$
$$H_1 = C\ ,$$
$$H_2 = D\ ,$$

we obtain

$$H_1(c_2 - c_0) + H_2(c_3 - c_1) = C(d - a) + D(b - c)$$

and, finally,

$$C(d - a) + D(b - c) = 2.$$

4.3. Normalization Algorithm

Nota bene: The obtained normalization condition is one condition only for

$$n + (n + 2) = 2n + 2 = 6$$

unknowns

$$H_i \ (i = 1, 2),$$
$$c_i \ (i = 0, 1, 2, 3).$$

Additionally,

$$H_i \geq 0 \ (i = 1, 2),$$
$$c_0 \leq c_1 \leq c_2 \leq c_3 \ .$$

Generally, it is not possible to simply take any admissible values of

$$2n + 2 - 1 = 2n + 1 = 5$$

unknowns and then to determine the value of the remaining unknown via this condition because it can happen that this value is inadmissible.
A natural idea, way, and algorithm to avoid this difficulty are as follows:
1. Fix

$$c_0 \leq c_1 \leq c_2 \leq c_3 \ .$$

2. Take any

$$H_i' \geq 0 \ (i = 1, 2)$$

so that there is at least one namely positive number among these $n = 2$ non-negative numbers.
3. Let

$$H_i \ (i = 1, 2)$$

be proportional to

$$H_i' \geq 0 \ (i = 1, 2),$$

respectively, with a common namely positive factor k so that

$$H_i = kH_i' \ (i = 1, 2).$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^2 H_i(c_{i+1} - c_{i-1}) = 2$$

so that

$$k = 2 / \Sigma_{i=0}^2 H_i'(c_{i+1} - c_{i-1}).$$

5. Explicitly determine

$$H_i = kH_i' \ (i = 1, 2).$$

Using

$$c_0 = a \ ,$$
$$c_1 = c \ ,$$
$$c_2 = d \ ,$$
$$c_3 = b \ ,$$
$$H_1 = C \ ,$$
$$H_2 = D$$

and naturally denoting

$$H_1' = C' \ ,$$
$$H_2' = D' \ ,$$

we obtain the same algorithm in the following form:
1. Fix

$$a \leq c \leq d \leq b \ .$$

2. Take any

$$C' \geq 0,$$
$$D' \geq 0$$

so that there is at least one namely positive number among these $n = 2$ non-negative numbers.
3. Let C and D be proportional to C' and D', respectively, with a common namely positive factor k so that

$$C = kC' ,$$
$$D = kD' .$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$C(d - a) + D(b - c) = 2$$

so that

$$k = 2/[C'(d - a) + D'(b - c)].$$

5. Explicitly determine

$$C = kC' ,$$
$$D = kD' .$$

We may also modify the same algorithm as follows:
1. Fix

$$a \leq c \leq d \leq b .$$

2. Take any

$$C' \geq 0,$$
$$D' \geq 0$$

so that there is at least one namely positive number among these $n = 2$ non-negative numbers.
3. Divide C' and D' by C' + D' to provide namely the unit sum of $n = 2$ non-negative numbers

$$w = C'/(C' + D'),$$
$$1 - w = D'/(C' + D').$$

4. Let C and D be proportional to w and 1 - w , respectively, with a common namely positive factor k so that

$$C = kw ,$$
$$D = k(1 - w).$$

5. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$C(d - a) + D(b - c) = 2$$

so that

$$k = 2/[w(d - a) + (1 - w)(b - c)].$$

6. Explicitly determine

$$C = 2w/[w(d - a) + (1 - w)(b - c)],$$
$$D = 2(1 - w)/[w(d - a) + (1 - w)(b - c)].$$

4.4. Mean Value (Mathematical Expectation)

Take the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx .$$

Use the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 2 we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6$$
$$= \Sigma_{i=1}^{2} H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Using

$$c_0 = a ,$$
$$c_1 = c ,$$
$$c_2 = d ,$$
$$c_3 = b ,$$
$$H_1 = C ,$$
$$H_2 = D ,$$

we obtain the same formula in the following form:

$$\mu = [H_1(c_2 - c_0)(c_2 + c_1 + c_0) + H_2(c_3 - c_1)(c_3 + c_2 + c_1)]/6,$$
$$\mu = [C(d - a)(d + c + a) + D(b - c)(b + d + c)]/6,$$
$$\mu = [C(d - a)(a + c + d) + D(b - c)(b + c + d)]/6.$$

Using

$$C = 2w/[w(d - a) + (1 - w)(b - c)],$$
$$D = 2(1 - w)/[w(d - a) + (1 - w)(b -c)],$$

finally determine

$$\mu = 1/3\ [w(d - a)(a + c + d) + (1 - w)(b - c)(b + c + d)]/[w(d - a) + (1 - w)(b - c)].$$

To compare this result with the corresponding formula [Dorp Kotz] for 1 - w > 0, denote

$$\alpha = w/(1 - w)$$

and obtain

$$\mu = 1/3\ [\alpha(d - a)(a + c + d) + (b - c)(b + c + d)]/[\alpha(d - a) + b - c].$$

4.5. Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \leq \nu) \geq 1/2$$

and

$$P(X \geq \nu) \geq 1/2.$$

For a continual real-number random variable X ,

$$P(X \leq \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \geq \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the same natural idea, way, and algorithm as for a general one-dimensional piecewise linear probability density but, naturally, with the formulas for a tetragonal probability density.

But using $n = 2$, make the same natural idea, way, and algorithm much more explicit:

1. First determine both

$$F(c) = \int_{-\infty}^{c} f(x)dx = \int_{a}^{c} f(x)dx = \int_{a}^{c} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{c}(x - a)dx = C/(c - a)\, [(c^2 - a^2)/2 - a(c - a)]$$
$$= C[(c + a)/2 - a] = C(c - a)/2$$

and

$$F(d) = 1 - \int_{d}^{+\infty} f(x)dx = 1 - \int_{d}^{b} f(x)dx = 1 - \int_{d}^{b} D(b - x)/(b - d)\, dx$$
$$= 1 - D/(b - d) \int_{d}^{b}(b - x)dx = 1 - D/(b - d)\, [b(b - d) - (b^2 - d^2)/2]$$
$$= 1 - D[b - (b + d)/2] = 1 - D(b - d)/2.$$

2. If

$$F(c) > 1/2,$$

or, equivalently,

$$C(c - a) > 1,$$

then there is the only median value ν strictly between a and c so that

$$F(\nu) = 1/2,$$
$$F(\nu) = \int_{-\infty}^{\nu} f(x)dx = \int_{a}^{\nu} f(x)dx = \int_{a}^{\nu} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{\nu}(x - a)dx = C/(c - a)\, [(\nu^2 - a^2)/2 - a(\nu - a)]$$
$$= C/(c - a)\, (\nu - a)^2/2 = 1/2,$$
$$(\nu - a)^2 = (c - a)/C\, ,$$
$$\nu = a + [(c - a)/C]^{1/2}\, .$$

3. If

$$F(c) = 1/2,$$

or, equivalently,

$$C(c - a) = 1,$$

then there is the only median value

$$\nu = c\, .$$

4. If

$$F(d) < 1/2,$$

or, equivalently,

$$1 - D(b - d)/2 < 1/2,$$
$$D(b - d) > 1,$$

then there is the only median value ν strictly between d and b so that

$$F(\nu) = 1/2,$$
$$F(\nu) = 1 - \int_{\nu}^{+\infty} f(x)dx = 1 - \int_{\nu}^{b} f(x)dx = 1 - \int_{\nu}^{b} D(b - x)/(b - d)\, dx$$
$$= 1 - D/(b - d) \int_{\nu}^{b}(b - x)dx = 1 - D/(b - d)\, [b(b - \nu) - (b^2 - \nu^2)/2]$$
$$= 1 - D/(b - d)\, (b - \nu)^2/2 = 1/2,$$
$$D/(b - d)\, (b - \nu)^2 = 1,$$
$$(b - \nu)^2 = (b - d)/D\, ,$$

$$\nu = b - [(b - d)/D]^{1/2} .$$

5. If

$$F(d) = 1/2,$$

or, equivalently,

$$1 - D(b - d)/2 = 1/2,$$
$$D(b - d) = 1,$$

then there is the only median value

$$\nu = d .$$

6. Finally, if

$$F(c) < 1/2 < F(d),$$

or, equivalently,

$$C(c - a) < 1$$

and

$$D(b - d) < 1,$$

then there is the only median value ν strictly between c and d (c < ν < d) because incremental distribution function F(c) strictly monotonically increases on this interval (c , d) so that

$$F(\nu) = 1/2,$$
$$F(\nu) = \int_{-\infty}^{\nu} f(x)dx = \int_a^{\nu} f(x)dx = \int_a^c f(x)dx + \int_c^{\nu} f(x)dx$$
$$= F(c) + \int_c^{\nu} [C(d - x) + D(x - c)]/(d - c)\, dx$$
$$= C(c - a)/2 + \{C[d(\nu - c) - (\nu^2 - c^2)/2] + D[(\nu^2 - c^2)/2 - c(\nu - c)]\}/(d - c)$$
$$= C(c - a)/2 + [(Cd - Dc)(\nu - c) + (D - C)(\nu^2 - c^2)/2]/(d - c) = 1/2,$$
$$C(c - a)(d - c) + 2(Cd - Dc)(\nu - c) + (D - C)(\nu^2 - c^2) = d - c ,$$
$$(D - C)\nu^2 + 2(Cd - Dc)\nu + C(c - a)(d - c) - 2(Cd - Dc)c - (D - C)c^2 + c - d = 0.$$

6.1. If D = C and, naturally, positive, then

$$2C(d - c)\nu + C(c - a)(d - c) - 2C(d - c)c + c - d = 0,$$
$$2C\nu = 1 + C(a + c),$$
$$\nu = 1/(2C) + (a + c)/2.$$

Directly moving from left to right, we also obtain the same result

$$\nu = c + [1/2 - C(c - a)/2]/C$$

at once. We have

$$\nu - c = 1/(2C) + (a - c)/2 > 0$$

because

$$C(c - a) < 1.$$

Directly moving from right to left, we obtain

$$\nu = d - [1/2 - C(b - d)/2]/C = - 1/(2C) + d + (b - d)/2 = (b + d)/2 - 1/(2C)$$

at once. We have

$$d - \nu = d + 1/(2C) - (b + d)/2 > 0$$

because

$$C(b - d) < 1.$$

To prove the equivalence of these both formulas

$$\nu = 1/(2C) + (a + c)/2$$

and

$$\nu = (b + d)/2 - 1/(2C)$$

for ν , note that

$$1/(2C) + (a + c)/2 = (b + d)/2 - 1/(2C)$$

because the normalization condition

$$C(c - a)/2 + C(d - c) + C(b - d)/2 = 1$$

gives

$$(b - a + d - c)/2 = 1/C .$$

6.2. If D ≠ C , then there is the only median value ν strictly between c and d (c < ν < d) because incremental distribution function F(c) strictly monotonically increases on this interval (c , d) so that

$$F(\nu) = 1/2.$$

Hence quadratic equation

$$(D - C)\nu^2 + 2(Cd - Dc)\nu + C(c - a)(d - c) - 2(Cd - Dc)c - (D - C)c^2 + c - d = 0$$

in ν has exactly one solution on this interval (c , d).

4.6. Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density function f(x) takes its maximum value f_{max} .
If C = D and, naturally, positive, then there are two modes c and d .
If C > D , then there is the only mode c .
If C < D , then there is the only mode d .

4.7. Variance

Take the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

Use the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 2 we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx$$
$$= \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})[c_{i+1}^2 + c_i^2 + c_{i-1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_ic_{i-1} - 4\mu(c_{i+1} + c_i + c_{i-1}) + 6\mu^2]/12$$
$$= \Sigma_{i=1}^2 H_i(c_{i+1} - c_{i-1})[c_{i+1}^2 + c_i^2 + c_{i-1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_ic_{i-1} - 4\mu(c_{i+1} + c_i + c_{i-1}) + 6\mu^2]/12$$

where

$$\mu = \Sigma_{i=1}^2 H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Using

$$c_0 = a ,$$
$$c_1 = c ,$$
$$c_2 = d ,$$
$$c_3 = b ,$$
$$H_1 = C ,$$
$$H_2 = D ,$$

we obtain the same formulas in the following forms:

$$\mu = [H_1(c_2 - c_0)(c_2 + c_1 + c_0) + H_2(c_3 - c_1)(c_3 + c_2 + c_1)]/6,$$
$$\mu = [C(d - a)(a + c + d) + D(b - c)(b + c + d)]/6,$$

as well as

$$\sigma^2 = H_1(c_2 - c_0)[c_2^2 + c_1^2 + c_0^2 + c_2c_1 + c_2c_0 + c_1c_0 - 4\mu(c_2 + c_1 + c_0) + 6\mu^2]$$
$$+ H_2(c_3 - c_1)[c_3^2 + c_2^2 + c_1^2 + c_3c_2 + c_3c_1 + c_2c_1 - 4\mu(c_3 + c_2 + c_1) + 6\mu^2]\}/12$$
$$= \{C(d - a)[d^2 + c^2 + a^2 + dc + da + ca - 4\mu(d + c + a) + 6\mu^2]$$
$$+ D(b - c)[b^2 + d^2 + c^2 + bd + bc + dc - 4\mu(b + d + c) + 6\mu^2]\}/12.$$

Finally,

$$\sigma^2 = \{C(d - a)[a^2 + c^2 + d^2 + ac + ad + cd - 4\mu(a + c + d) + 6\mu^2]$$
$$+ D(b - c)[b^2 + c^2 + d^2 + bc + bd + cd - 4\mu(b + c + d) + 6\mu^2]\}/12.$$

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

5. Piecewise Linear Probability Density Formulas Verification via a Triangular Probability Distribution

5.1. Main Definitions

Verify formulas for a general one-dimensional piecewise linear probability density using formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution as a particular case of a general one-dimensional piecewise linear continuous probability density for n = 1 and further of a general one-dimensional piecewise linear probability density. Therefore, directly apply the above formulas for a general one-dimensional piecewise linear continuous probability density (or, alternatively, for a tetragonal probability density) to a triangular probability density (Fig. 4).

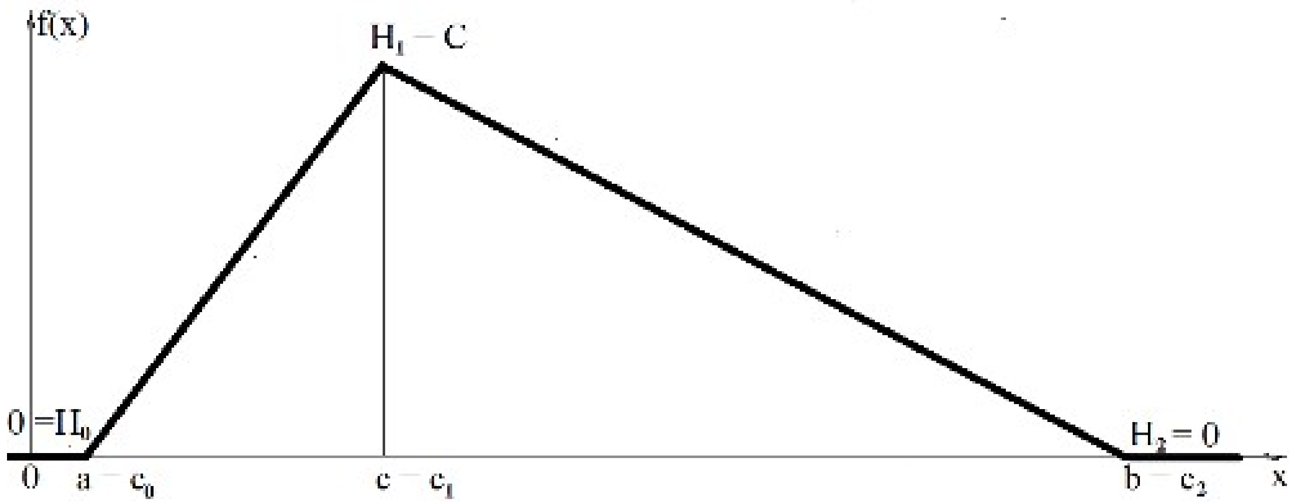

Figure 6. Triangular probability density

Here probability density function f(x) is as always non-negative everywhere ($-\infty < x < +\infty$) and can be positive on some finite segment (closed interval)

$$-\infty < a \le x \le b < +\infty \ (a < b)$$

only. Let n = 1 intermediate point $c = c_1$ so that

$$a \le c_1 \le b$$

divide this segment into n + 1 = 2 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a \ ,$$

$$c_2 = b \ ,$$

$$c(i) = c_i \ (i = 0, 1, 2).$$

On each of n + 1 = 2 closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1),$$

probability density function f(x) is linear. At n + 2 = 3 points

$$c_i \ (i = 0, 1, 2),$$

f(x) takes finite non-negative values

$$H_i = f(c_i),$$

respectively. Naturally, we have

$$H_0 = 0,$$
$$H_2 = 0.$$

Note that

$$H_1 = f(c_1)$$

with additional natural notation

$$C = H_1$$

for value f(x) at point

$$c = c_1$$

may be any finite positive value. At each of n + 2 = 3 points

$$c_i \ (i = 0, 1, 2),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are equal to one another and coincide with $f(c_i)$. Therefore, we obtain

$$H_i = L_i = R_i \ (i = 0, 1, 2).$$

Then on each of n + 1 = 2 closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1),$$

linear probability density function

$$f(x) = H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= [H_i(c_{i+1} - x) + H_{i+1}(x - c_i)]/(c_{i+1} - c_i).$$

Use short (reduced) notation [Gelimson 2012a] and the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 1 we represent non-negative-valued function f(x) via

$$f_{[0\,,\,\infty)}(x_{(-\infty\,,\,\infty)}) = 0_{(-\infty\,,\,c(0))\,\cup\,[c(3),\,\infty)} \cup \cup_{i\,=\,0}{}^1\, [H_i(c_{i+1} - x) + H_{i+1}(x - c_i)]/(c_{i+1} - c_i)_{[c(i),\,c(i+1))}$$

on the whole real axis $(-\infty , \infty)$.

Using

$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = b\ ,$$
$$H_0 = 0,$$
$$H_1 = C\ ,$$
$$H_2 = 0,$$

we obtain

$$f_{[0\,,\,\infty)}(x_{(-\infty\,,\,\infty)}) = 0_{(-\infty\,,\,c(0))\,\cup\,[c(2),\,\infty)} \cup [H_0(c_1 - x) + H_1(x - c_0)]/(c_1 - c_0)\}_{[c(0),\,c(1))}$$
$$\cup\ [H_1(c_2 - x) + H_2(x - c_1)]/(c_2 - c_1)\}_{[c(1),\,c(2))}\ ,$$
$$f_{[0\,,\,\infty)}(x_{(-\,\infty,\,\infty)}) = 0_{(-\infty\,,\,a)\cup[b\,,\,\infty)} \cup C(x - a)/(c - a)_{[a\,,\,c)} \cup C(b - x)/(b - c)_{[c\,,\,b)}\ .$$

Integral (cumulative) probability distribution function

$$F(x) = P(X \le x) = \int_{-\infty}{}^x f(t)dt$$

is probability $P(X \le x)$ that real-number random variable X takes a real-number value not greater than x .

5.2. Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

Use the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 1 we determine

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$

$$= \Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1})/2 = \Sigma_{i=1}^{1} H_i(c_{i+1} - c_{i-1})/2 = H_1(c_2 - c_0).$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the 2 rectangular triangles.

Therefore, to provide a possible (an admissible) probability density function, necessary and sufficient integral normalization condition

$$H_1(c_2 - c_0) = 2$$

has to be satisfied.

Using

$$c_0 = a\ ,$$

$$c_1 = c\ ,$$

$$c_2 = b\ ,$$

$$H_1 = C\ ,$$

we obtain

$$H_1(c_2 - c_0) = C(b - a)$$

and, finally,

$$C(b - a) = 2,$$

$$C = 2/(b - a).$$

The known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

5.3. Mean Value (Mathematical Expectation)

Take the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx\ .$$

Use the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 1 we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6$$
$$= \Sigma_{i=1}^{1} H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6$$

and, finally,

$$\mu = H_1(c_2 - c_0)(c_2 + c_1 + c_0)/6.$$

Using

$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = b\ ,$$
$$H_1 = C\ ,$$

we obtain the same formula in the following form:

$$\mu = C(b - a)(b + c + a)/6.$$

Using

$$C = 2/(b - a),$$

finally obtain

$$\mu = (a + b + c)/3.$$

The known formulas [Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

5.4. Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \le \nu) \ge 1/2$$

and

$$P(X \ge \nu) \ge 1/2.$$

For a continual real-number random variable X ,

$$P(X \le \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \ge \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the same natural idea, way, and algorithm as for a general one-dimensional piecewise linear probability density but, naturally, with the formulas for a triangular probability density.

But using n = 1, as well as the corresponding algorithm and formulas for a tetragonal probability density with

$$d = c ,$$
$$D = C ,$$
$$C = 2/(b - a),$$

make the same natural idea, way, and algorithm as for a general one-dimensional piecewise linear probability density much more explicit:

1. First determine

$$F(c) = \int_{-\infty}^{c} f(x)dx = \int_{a}^{c} f(x)dx = \int_{a}^{c} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{c}(x - a)dx = C/(c - a)\, [(c^2 - a^2)/2 - a(c - a)]$$
$$= C[(c + a)/2 - a] = C(c - a)/2 = (c - a)/(b - a).$$

2. If

$$F(c) > 1/2,$$

or, equivalently,

$$c > (a + b)/2,$$

then there is the only median value ν strictly between a and c so that

$$F(\nu) = 1/2,$$
$$F(\nu) = \int_{-\infty}^{\nu} f(x)dx = \int_{a}^{\nu} f(x)dx = \int_{a}^{\nu} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{\nu}(x - a)dx = C/(c - a)\, [(\nu^2 - a^2)/2 - a(\nu - a)]$$
$$= C/(c - a)\, (\nu - a)^2/2 = 1/2,$$
$$(\nu - a)^2 = (c - a)/C ,$$
$$\nu = a + [(c - a)/C]^{1/2} ,$$
$$\nu = a + [(b - a)(c - a)/2]^{1/2} .$$

The known formulas [Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

3. If

$$F(c) = 1/2,$$

or, equivalently,

$$c = (a + b)/2,$$

then there is the only median value

$$\nu = c = (a + b)/2.$$

Naturally, the known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same obvious result.

4. If

$$F(c) < 1/2,$$

or, equivalently,

$$c < (a + b)/2,$$

then there is the only median value ν strictly between c and b so that

$$F(\nu) = 1/2,$$
$$F(\nu) = 1 - \int_\nu^{+\infty} f(x)dx = 1 - \int_\nu^b f(x)dx = 1 - \int_\nu^b C(b - x)/(b - c)\, dx$$
$$= 1 - C/(b - c) \int_\nu^b (b - x)dx = 1 - C/(b - c)\, [b(b - \nu) - (b^2 - \nu^2)/2]$$
$$= 1 - C/(b - c)\, (b - \nu)^2/2 = 1/2,$$
$$C/(b - c)\, (b - \nu)^2 = 1,$$
$$(b - \nu)^2 = (b - c)/C\ ,$$
$$\nu = b - [(b - c)/C]^{1/2}$$
$$\nu = b - [(b - a)(b - c)/2]^{1/2}\ .$$

The known formulas [Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

These three conditional formulas for the only median value ν can be unified as follows:

$$\nu = (a + b)/2 + \{[(b - a)(b - a + |2c - a - b|)]^{1/2} + a - b\}/2\ \mathrm{sign}(2c - a - b).$$

In fact, we obtain:

1) by $c > (a + b)/2$,

$$\nu = (a + b)/2 + \{[(b - a)(b - a + 2c - a - b)]^{1/2} + a - b\}/2$$
$$= (a + b)/2 + \{[(b - a)(2c - 2a)]^{1/2} + a - b\}/2$$
$$= a + [(b - a)(c - a)/2]^{1/2}\ ;$$

2) by $c = (a + b)/2$,

$$\nu = (a + b)/2;$$

3) by $c < (a + b)/2$,

$$\nu = (a + b)/2 - \{[(b - a)(b - a - 2c + a + b)]^{1/2} + a - b\}/2$$
$$= (a + b)/2 - \{[(b - a)(2b - 2c)]^{1/2} + a - b\}/2$$
$$= b - [(b - a)(b - c)/2]^{1/2}\ .$$

5.5. Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density function f(x) takes its maximum value f_{max} .
In our case, there is the only mode c .
Naturally, the known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same obvious result.

5.6. Variance

Take the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

Use the corresponding formula for a piecewise linear continuous probability density. Then in our case n = 1 we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx$$

$$= \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})[c_{i+1}^2 + c_i^2 + c_{i-1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_ic_{i-1} - 4\mu(c_{i+1} + c_i + c_{i-1}) + 6\mu^2]/12$$

$$= \Sigma_{i=1}^1 H_i(c_{i+1} - c_{i-1})[c_{i+1}^2 + c_i^2 + c_{i-1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_ic_{i-1} - 4\mu(c_{i+1} + c_i + c_{i-1}) + 6\mu^2]/12$$

and, finally,

$$\sigma^2 = H_1(c_2 - c_0)[c_2^2 + c_1^2 + c_0^2 + c_2c_1 + c_2c_0 + c_1c_0 - 4\mu(c_2 + c_1 + c_0) + 6\mu^2]/12$$

where

$$\mu = \Sigma_{i=1}^1 H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6 = H_1(c_2 - c_0)(c_2 + c_1 + c_0)/6.$$

Using

$$c_0 = a ,$$
$$c_1 = c ,$$
$$c_2 = b ,$$
$$H_1 = C ,$$
$$C = 2/(b - a),$$

or, alternatively, the above formulas for a tetragonal probability density with

$$d = c ,$$
$$D = C ,$$

we obtain the same formulas in the following forms:

$$\mu = C(b - a)(a + b + c)/6,$$
$$\mu = (a + b + c)/3,$$

as well as

$$\sigma^2 = C(b - a)[b^2 + c^2 + a^2 + bc + ba + ca - 4\mu(b + c + a) + 6\mu^2]/12,$$
$$\sigma^2 = [a^2 + b^2 + c^2 + ab + ac + bc - 4\mu(a + b + c) + 6\mu^2]/6,$$

Substituting

$$\mu = (a + b + c)/3,$$

we obtain

$$\sigma^2 = [a^2 + b^2 + c^2 + ab + ac + bc - 4/3\ (a + b + c)^2 + 2/3\ (a + b + c)^2]/6,$$
$$\sigma^2 = [3(a^2 + b^2 + c^2 + ab + ac + bc) - 2(a + b + c)^2]/18,$$
$$\sigma^2 = (3a^2 + 3b^2 + 3c^2 + 3ab + 3ac + 3bc - 2a^2 - 2b^2 - 2c^2 - 4ab - 4ac - 4bc)/18,$$
$$\sigma^2 = (a^2 + b^2 + c^2 - ab - ac - bc)/18.$$

Alternatively,

$$\sigma^2 = [(c - a)^2 + (b - c)^2 + (b - a)^2]/36.$$

The known formulas [Kotz & van Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

Main Results and Conclusions

1. A piecewise linear probability density is very simple, natural, and typical, as well as sufficiently general.
2. A general one-dimensional piecewise linear probability density is very suitable for adequately modeling via efficiently approximating practically arbitrary nonlinear probability density with any required precision.
3. The explicit normalization, expectation, and variance formulas along with the median and mode formulas and algorithms for a general one-dimensional piecewise linear probability density are obtained and developed.
4. These formulas and algorithms are also applied to a general one-dimensional piecewise linear continuous probability density.
5. The formulas and algorithms for a general one-dimensional piecewise linear continuous probability density are very suitable for its important particular case, namely for a tetragonal probability density. It is also a natural generalization of a triangular probability density.
6. The known formulas for a triangular probability density as a further particular case of a general one-dimensional piecewise linear probability density provide verifying the obtained formulas and algorithms.
7. To additionally verify the present analytical methods, geometrical approach can be also applied if possible and useful.
8. The problems of the existence and uniqueness of the mean, median, and mode values for a general one-dimensional piecewise linear probability density are set and algorithmically solved.
9. The obtained formulas and developed algorithms have clear mathematical (probabilistic and statistical) sense and are simple and very suitable for setting and solving many typical urgent problems.
10. Piecewise linear probability density theory provides scientific basis for discovering and thoroughly investigating many complex phenomena and relations not only in probability theory and mathematical statistics, but also in physics, engineering, chemistry, biology, medicine, geology, astronomy, meteorology, agriculture, politics, management, economics, finance, psychology, etc.

Bibliography

[Cantor] Georg Cantor. Gesammelte Abhandlungen mathematischen und philosophischen Inhalts. Springer, Berlin 1932

[Cramér] Harald Cramér. Mathematical Methods of Statistics. Princeton University Press, 1945

[Daniell] Percy John Daniell. A General Form of Integral. Annals of Mathematics, Second Series, 1918, 19(4), pp. 279-294

[Darboux] Gaston Darboux. Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Vols. 1 to 4. Gauthier-Villars, 1887-96

[Denjoy 1912a] Arnaud Denjoy. Une extension de l'integrale de M. Lebesgue. C. R. Acad. Sci., 154 (1912), pp. 859-862

[Denjoy 1912b] Arnaud Denjoy. Calcul de la primitive de la fonction dérivée la plus générale. C. R. Acad. Sci., 154 (1912), pp. 1075-1078

[Denjoy 1916] Arnaud Denjoy. Sur la dérivation et son calcul inverse. C. R. Acad. Sci., 162 (1916), pp. 377-380

[van Dorp & Kotz] J. René van Dorp, Samuel Kotz. Generalized Trapezoidal Distributions. Metrika, Vol. 58, Issue 1, July 2003, 85-97. DOI:10.1007/s001840200230

[Encyclopaedia of Mathematics] Encyclopaedia of Mathematics. Ed. Michiel Hazewinkel. Volumes 1 to 10. Supplements I to III. Kluwer Academic Publ., Dordrecht, 1987-2002

[Gelimson 2003a] Lev Gelimson. Quantianalysis: Uninumbers, Quantioperations, Quantisets, and Multiquantities (now Uniquantities). Abhandlungen der WIGB (Wissenschaftlichen Gesellschaft zu Berlin), **3** (2003), Berlin, 15-21

[Gelimson 2003b] Lev Gelimson. General Problem Theory. Abhandlungen der WIGB (Wissenschaftlichen Gesellschaft zu Berlin), **3** (2003), Berlin, 26-32

[Halmos] Paul R. Halmos. Measure Theory. D. Van Nostrand Company, Inc., New York, N. Y., 1950

[Henstock] Ralph Henstock. Theories of Integration. Butterworths, London, 1963

[Karlis & Xekalaki] Dimitris Karlis, Evdokia Xekalaki. The Polygonal Distribution. International Conference on Mathematical and Statistical Modeling in Honor of Enrique Castillo, June 28-30, 2006. Advances in Mathematical and Statistical Modeling Statistics for Industry and Technology 2008, pp 21-33. http://www.uclm.es/actividades0506/congresos/icmsm2006/articles/KarlisX06.pdf

[Khintchine] Alexandre Khintchine. Sur une extension de l'integrale de M. Denjoy. C. R. Acad. Sci., 162 (1916), pp. 287-291

[Kim] Jay J. Kim. Application of the Truncated Triangular and the Trapezoidal Distributions. Proceedings of the Survey Research Methods Section, American Statistical Association (2011). Session 191: Synthetic Data and Other Methods for Disclosure Limitation and Confidentiality Preservation. http://www.amstat.org/sections/srms/Proceedings/y2007/Files/JSM2007-000166.pdf

[Kolmogorov & Fomin] Andrey N. Kolmogorov, Sergey V. Fomin. Introductory Real Analysis. Dover Publications, New York, 1970

[Kotz & van Dorp] Samuel Kotz, Johan René van Dorp. Beyond Beta: Other Continuous Families of Distributions with Bounded Support and Applications. World Scientific, 2004. Chapter 1: The

Triangular Distribution. http://books.google.de/books?id=NpK1xSB_SE4C http://www.worldscientific.com/doi/suppl/10.1142/5720/suppl_file/5720_chap1.pdf

[Kurzweil] Jaroslav Kurzweil. Generalized ordinary differential equations and continuous dependence on a parameter. (Russian) Czechoslovak Math., J. 7 (82), 1957, pp. 418-449

[Lebesgue] Henri Lebesgue. Leçons sur l'intégration et la recherche des fonctions primitives. Gauthier-Villars, Paris, 1904. 2nd Edition, 1928.

[Loève] Michel Loève. Probability Theory. Van Nostrand, Princeton (New Jersey), 1955

[Lusin] Nicolas Lusin. Leçons sur les Ensembles Analytiques et leurs Applications. With a preface by Henri Lebesgue and a note by Waclaw Sierpinski. Gauthier-Villars, Paris, 1930

[Natanson] Isidor P. Natanson. Theory of functions of a real variable. Frederick Ungar Publishing Co., New York, 1955

[Perron] Oskar Perron. Über den Integralbegriff. Sitzungsber. Heidelberg. Akad. Wiss., VA (1914), pp. 1-16

[Radon] Johann Radon. Collected Works. Volumes 1, 2. Birkhäuser-Verlag, Basel-Boston, 1987

[Riemann] Bernhard Riemann. Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe (On the representability of a function by a trigonometric series). University of Göttingen, Habilitationsschrift, 1854. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen (Proceedings of the Royal Philosophical Society at Göttingen), 1868, vol. 13, pp. 87-132

[Saks] Stanisław Saks. Theory of the Integral. Monografie Matematyczne 7 (2nd ed.). G. E. Stechert & Co., Warszawa-Lwów, 1937. English translation by Laurence Chisholm Young, with two additional notes by Stefan Banach

[Shilov & Gurevich] Georgy E. Shilov, Boris L. Gurevich. Integral, Measure, and Derivative: A Unified Approach. Richard A. Silverman, trans. Dover Publications, 1978

[Smith] Henry J. S. Smith. On the integration of discontinuous functions. Proceedings of the London Mathematical Society, 1874, Series 1, vol. 6, pp. 140-153.

[Stieltjes] Thomas Jan Stieltjes. Recherches sur les fractions continues (Researches on continued fractions". Annales de la Faculte de Sciences de Toulouse, 8(1), 1894, pp. 1-122

[Wikipedia Triangular distribution] http://en.wikipedia.org/wiki/Triangular_distribution