Piecewise Linear Probability Distribution

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Abstract

Introduction

Both particular and some more general mostly continuous (continual without discontinuity points with jumps) piecewise linear probability distributions which can also be multidimensional are well known [Cramér]. For a triangular probability distribution, some basic formulas are also well known [Kotz Dorp, Wikipedia Triangular distribution]. The present work is dedicated to obtaining some more general formulas for more general piecewise linear probability distributions.

Piecewise Linear Probability Distribution

Main Definitions

Consider a general one-dimensional piecewise linear probability distribution (Fig. 1).

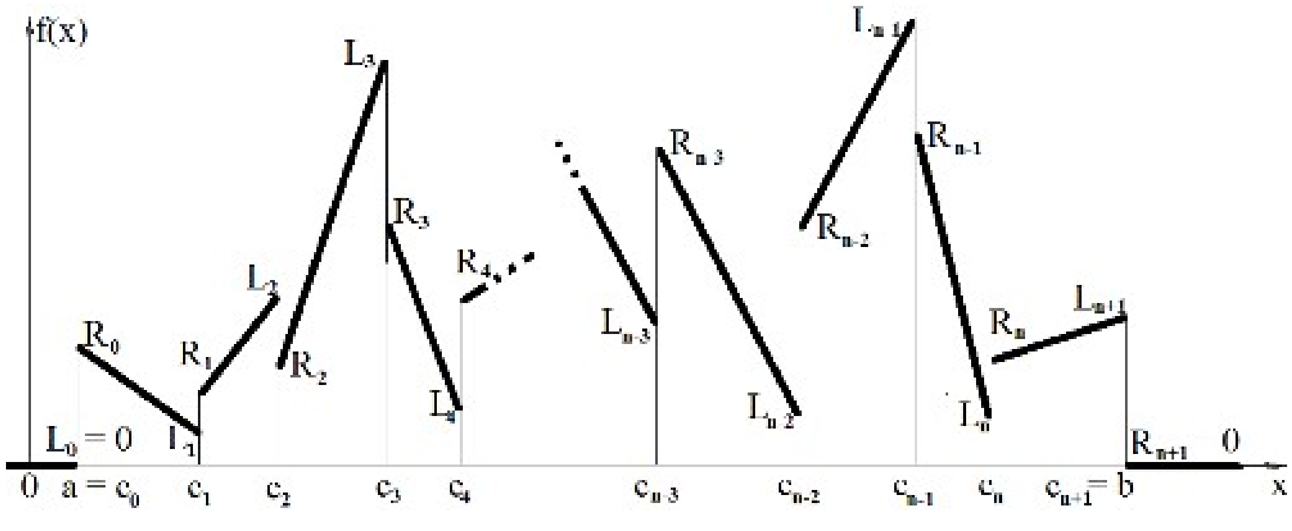

Fig. 1. General one-dimensional piecewise linear probability distribution

Here probability density distribution function f(x) is as always non-negative everywhere ($-\infty < x < +\infty$) and can be positive on some finite segment (closed interval)

$$-\infty < a \le x \le b < +\infty \ (a < b)$$

only. Let n ($n \in N = \{1, 2, ...\}$) intermediate points c_1 , c_2 , c_3 , c_4 , … , c_{n-3} , c_{n-2} , c_{n-1} , c_n in the non-decreasing order so that

$$a \le c_1 \le c_2 \le c_3 \le c_4 \le \ldots \le c_{n-3} \le c_{n-2} \le c_{n-1} \le c_n \le b$$

divide this segment into n + 1 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a ,$$
$$c_{n+1} = b ,$$
$$c(i) = c_i \ (i = 0, 1, 2, \ldots , n + 1).$$

On each of n + 1 open intervals

$$c_i < x < c_{i+1} \ (i = 0, 1, 2, \ldots , n),$$

probability density distribution function f(x) is linear. At n + 2 points

$$c_i \ (i = 0, 1, 2, \ldots , n + 1),$$

f(x) may take any finite real values. The following considerations (possibly excepting mode values below) do not depend on these values. At each of n + 2 points

$$c_i \ (i = 0, 1, 2, \ldots , n + 1),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are any generally different finite real values. Naturally, we have

$$L_0 = 0,$$
$$R_{n+1} = 0.$$

Then on each of n + 1 open intervals

$$c_i < x < c_{i+1} \ (i = 0, 1, 2, \ldots , n),$$

linear probability density distribution function

$$f(x) = R_i + (L_{i+1} - R_i)(x - c_i)/(c_{i+1} - c_i).$$

Integral (cumulative) probability distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

is probability $P(X \le x)$ that real-number random variable X takes a real-number value not greater than x .

Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

In our case we have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$
$$= \Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} f(x)dx = \Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} [R_i + (L_{i+1} - R_i)(x - c_i)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^{n} \{R_i(c_{i+1} - c_i) + (L_{i+1} - R_i)[(c_{i+1}^2 - c_i^2)/2 - c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$
$$= \Sigma_{i=0}^{n} [R_i(c_{i+1} - c_i) + (L_{i+1} - R_i)(c_{i+1} - c_i)/2]$$
$$= \Sigma_{i=0}^{n} (R_i + L_{i+1})(c_{i+1} - c_i)/2.$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the n + 1 rectangular trapezoids.

Therefore, to provide a possible (an admissible) probability density distribution function, necessary and sufficient integral normalization condition

$$\Sigma_{i=0}^{n} (R_i + L_{i+1})(c_{i+1} - c_i) = 2$$

has to be satisfied.

Normalization Algorithm

Nota bene: This is one condition for

$$(n + 1) + (n + 1) + (n + 2) = 3n + 4$$

unknowns

$$R_i \ (i = 0, 1, 2, \dots , n),$$
$$L_i \ (i = 1, 2, 3, \dots , n + 1),$$
$$c_i \ (i = 0, 1, 2, \dots , n + 1).$$

Additionally,

$$R_i \geq 0 \ (i = 0, 1, 2, \dots , n),$$
$$L_i \geq 0 \ (i = 1, 2, 3, \dots , n + 1),$$
$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \dots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} .$$

Generally, it is not possible to simply take any admissible values of

$$3n + 4 - 1 = 3n + 3$$

unknowns and then to determine the value of the remaining unknown via this condition because it can happen that this value is inadmissible.

A natural idea, way, and algorithm to avoid this difficulty are as follows:

1. Fix

$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \dots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} .$$

2. Take any

$$R_i' \geq 0 \ (i = 0, 1, 2, \dots , n),$$
$$L_i' \geq 0 \ (i = 1, 2, 3, \dots , n + 1)$$

so that there is at least one namely positive number among these 2n + 2 non-negative numbers.

3. Let

$$R_i \ (i = 0, 1, 2, \dots , n),$$
$$L_i \ (i = 1, 2, 3, \dots , n + 1)$$

be proportional to

$$R_i' \geq 0 \ (i = 0, 1, 2, \dots , n),$$
$$L_i' \geq 0 \ (i = 1, 2, 3, \dots , n + 1),$$

respectively, with a common namely positive factor k so that

$$R_i = kR_i' \ (i = 0, 1, 2, \dots , n),$$
$$L_i = kL_i' \ (i = 1, 2, 3, \dots , n + 1).$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$\Sigma_{i=0}^n (R_i + L_{i+1})(c_{i+1} - c_i) = 2$$

so that

$$k = 2 / \Sigma_{i=0}^n (R_i' + L_{i+1}')(c_{i+1} - c_i).$$

5. Explicitly determine

$$R_i = kR_i' \ (i = 0, 1, 2, \dots , n),$$
$$L_i = kL_i' \ (i = 1, 2, 3, \dots , n + 1).$$

Mean Value (Mathematical Expectation)

Use the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx .$$

In our case we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} xf(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [R_ix + (L_{i+1} - R_i)(x^2 - c_ix)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^n \{R_i(c_{i+1}^2 - c_i^2)/2 + (L_{i+1} - R_i)[(c_{i+1}^3 - c_i^3)/3 - c_i(c_{i+1}^2 - c_i^2)/2]/(c_{i+1} - c_i)\}$$
$$= 1/6\ \Sigma_{i=0}^n \{3R_i(c_{i+1}^2 - c_i^2) + (L_{i+1} - R_i)[2(c_{i+1}^2 + c_{i+1}c_i + c_i^2) - 3c_i(c_{i+1} + c_i)]\}$$
$$= 1/6\ \Sigma_{i=0}^n [3R_i(c_{i+1}^2 - c_i^2) + (L_{i+1} - R_i)(2c_{i+1}^2 - c_ic_{i+1} - c_i^2)]$$
$$= 1/6\ \Sigma_{i=0}^n (c_{i+1} - c_i)[3R_i(c_{i+1} + c_i) + (L_{i+1} - R_i)(2c_{i+1} + c_i)]$$
$$= 1/6\ \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(c_{i+1} + 2c_i) + L_{i+1}(2c_{i+1} + c_i)]$$

and finally

$$\mu = \Sigma_{i=0}^{n} (c_{i+1} - c_i)[R_i(2c_i + c_{i+1}) + L_{i+1}(c_i + 2c_{i+1})]/6.$$

Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \le \nu) \ge 1/2$$

and

$$P(X \ge \nu) \ge 1/2.$$

For a continual real-number random variable X ,

$$P(X \le \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \ge \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the following natural idea, way, and algorithm:

1. First consider

$$c_i \ (i = 0, 1, 2, \dots , n + 1)$$

not far from μ and determine both

$$L = \max\{i \mid \int_{-\infty}^{c(i)} f(x)dx < 1/2\}$$

and

$$R = \min\{i \mid \int_{c(i)}^{+\infty} f(x)dx < 1/2\}.$$

Then both

$$\int_{-\infty}^{c(L+1)} f(x)dx \ge 1/2$$

and

$$\int_{c(R-1)}^{+\infty} f(x)dx \ge 1/2.$$

2. On half-closed interval

$$c(L) = c_L < \nu \le c_{L+1} = c(L+1),$$

determine

$$\nu_{min} = \inf\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}.$$

3. On half-closed interval

$$c(R-1) = c_{R-1} \le \nu < c_R = c(R),$$

determine

$$\nu_{max} = \sup\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\}.$$

4. Then the set of all the median values ν is the interval whose endpoints are

$$\nu_{min} \le \nu_{max}$$

each of wich is included into the interval if and only if the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and/or

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

respectively.

Notata bene:

1. If

$$\nu_{min} = \nu_{max} ,$$

then the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

hence the closed interval

$$\nu_{min} \le \nu \le \nu_{max}$$

contains the only median value

$$\nu = \nu_{min} = \nu_{max} .$$

2. If

$$\nu_{min} < \nu_{max} ,$$

then the integral of f(x) on the interval whose endpoints are ν_{min} and ν_{max} vanishes independently of their including or excluding. Hence on this interval, non-negative probability density distribution function f(x) also vanishes possibly excepting points whose set has zero measure (in our case, a finite set).

Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density distribution function f(x) takes its maximum value f_{max} . For continual distributions, generalize this definition in the following directions:

1. Replace the maximum value f_{max} with the supremum value f_{sup} which always exists. The reason is that it is possible (for piecewise linear probability distributions, too) that function f(x) is discontinuous and does not take the supremum value f_{sup} so that the maximum value f_{max} does not exist at all.
2. Extend the range of function f(x), i.e. the set of values function f(x) really (truly) takes, via all the limiting points of this set. Then the extended range is a closed set and contains, in particular, the supremum value f_{sup} .
3. Extend the domain of function f(x), i.e. the set of points at which function f(x) is properly defined, via all the limiting points of this set. Then the extended domain is a closed set which contains all its limiting points.
4. Admit modes to also correspond to the one-sided limits of function f(x) separately if necessary. This is important for discontinuous function f(x) with jumps.
5. At any interval endpoint c_i , along with the given value of $f(c_i)$, take into account the one-sided limits L_i and R_i of function f(x), eg any of the following reasonable options for value $f(c_i)$:

5.1. Take the given value of $f(c_i)$ itself.

5.2. Take

$$f(c_i) = \max\{L_i , R_i\}.$$

5.3. Take

$$f(c_i) = (L_i + R_i)/2.$$

6. At any interval endpoint c_i , along with c_i itself, take into account the one-sided limiting points c_i - 0 and c_i + 0 corresponding to one-sided limits L_i and R_i of function f(x), respectively, eg any of the following reasonable options for c_i :

6.1. Take the given value of c_i itself.

6.2. For modes, rather than c_i , consider

$$c_i - 0 \text{ if } L_i > R_i ,$$
$$c_i + 0 \text{ if } L_i < R_i ,$$

and quantiset [Gelimson 2003a, 2003b]

$$\{{}_{1/2}(c_i - 0),\ {}_{1/2}(c_i + 0)\}^\circ \text{ if } L_i = R_i .$$

This quantiset consists of two quantielements

$${}_{1/2}(c_i - 0),\ {}_{1/2}(c_i + 0)$$

with bases

$$c_i - 0,\ c_i + 0,$$

respectively.

Here each of elements c_i - 0 and c_i + 0 has quantity 1/2 so that the total unit quantity is equally divided between these both elements.

In particular, for a piecewise linear probability distribution with probability density function f(x), anyone of the following values can reasonably play the role of f_{sup} :

$$\max\{\max\{f(c_i) \mid i = 0, 1, 2, \dots , n + 1\}, \max\{L_i \mid i = 0, 1, \dots , n + 1\}, \max\{R_i \mid i = 0, 1, \dots , n + 1\}\},$$
$$\max\{\max\{f(c_i) \mid i = 0, 1, 2, \dots , n + 1\}, \max\{(L_i + R_i)/2 \mid i = 0, 1, 2, \dots , n + 1\}\},$$
$$\max\{\max\{L_i \mid i = 0, 1, 2, \dots , n + 1\}, \max\{R_i \mid i = 0, 1, 2, \dots , n + 1\}\},$$
$$\max\{(L_i + R_i)/2 \mid i = 0, 1, 2, \dots , n + 1\}.$$

If $f(c_i) = f_{sup}$ at some i , then c_i at this i is one of the modes.
If $L_i = f_{sup}$ at some i , then $c_i - 0$ at this i is one of the modes.
If $R_i = f_{sup}$ at some i , then $c_i + 0$ at this i is one of the modes.
If $(L_i + R_i)/2 = f_{sup}$ at some i , then quantiset

$$\{{}_{1/2}(c_i - 0),\ {}_{1/2}(c_i + 0)\}^\circ$$

at this i is one of the modes.

Nota bene: The set of all the modes contains the corresponding separate points c_i , as well as one-sided limits $c_i - 0$ and $c_i + 0$, and includes open intervals

$$c_i < x < c_{i+1} \ (i = 1, 2, \dots , n - 1)$$

for which

$$R_i = L_{i+1} = f_{sup} .$$

Variance

Use the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

In our case we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} (x - \mu)^2 f(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [R_i(x - \mu)^2 + (L_{i+1} - R_i)(x - \mu)^2(x - c_i)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} \{R_i(x^2 - 2\mu x + \mu^2) + (L_{i+1} - R_i)[x^3 - (2\mu + c_i)x^2 + (\mu^2 + 2\mu c_i)x - \mu^2 c_i]/(c_{i+1} - c_i)\}dx$$
$$= \Sigma_{i=0}^n \{R_i[(c_{i+1}^3 - c_i^3)/3 - 2\mu(c_{i+1}^2 - c_i^2)/2 + \mu^2(c_{i+1} - c_i)]$$
$$+ (L_{i+1} - R_i)[(c_{i+1}^4 - c_i^4)/4 - (2\mu + c_i)(c_{i+1}^3 - c_i^3)/3 + (\mu^2 + 2\mu c_i)(c_{i+1}^2 - c_i^2)/2 - \mu^2 c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$
$$= 1/12\ \Sigma_{i=0}^n \{R_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (L_{i+1} - R_i)[3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - (4c_i + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) + (12\mu c_i + 6\mu^2)(c_{i+1} + c_i) - 12\mu^2 c_i]\}$$
$$= 1/12\ \Sigma_{i=0}^n [R_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (L_{i+1} - R_i)(3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - 4c_{i+1}^2 c_i - 4c_{i+1}c_i^2 - 4c_i^3 - 8\mu c_{i+1}^2 - 8\mu c_{i+1}c_i - 8\mu c_i^2 + 12\mu c_{i+1}c_i + 12\mu c_i^2 + 6\mu^2 c_{i+1} + 6\mu^2 c_i - 12\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^n [R_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (L_{i+1} - R_i)(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^n [R_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i - 3c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 + c_i^3 + 8\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 4\mu c_i^2 - 6\mu^2 c_{i+1} + 6\mu^2 c_i)$$
$$+ L_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^n [R_i(c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 - 3c_i^3 + 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 2\mu^2 c_{i+1} + 2\mu^2 c_i)$$
$$+ L_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^n (c_{i+1} - c_i)\{R_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$
$$+ L_{i+1}[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]\}$$

and finally

$$\sigma^2 = 1/12\ \Sigma_{i=0}^n (c_{i+1} - c_i)\{R_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$
$$+ L_{i+1}[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]\}$$

where

$$\mu = \Sigma_{i=0}^n (c_{i+1} - c_i)[R_i(2c_i + c_{i+1}) + L_{i+1}(c_i + 2c_{i+1})]/6.$$

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

Piecewise Linear Continuous Probability Distribution

Main Definitions

Consider a general one-dimensional piecewise linear continuous probability distribution (Fig. 2) as a particular case of a general one-dimensional piecewise linear continuous probability distribution.

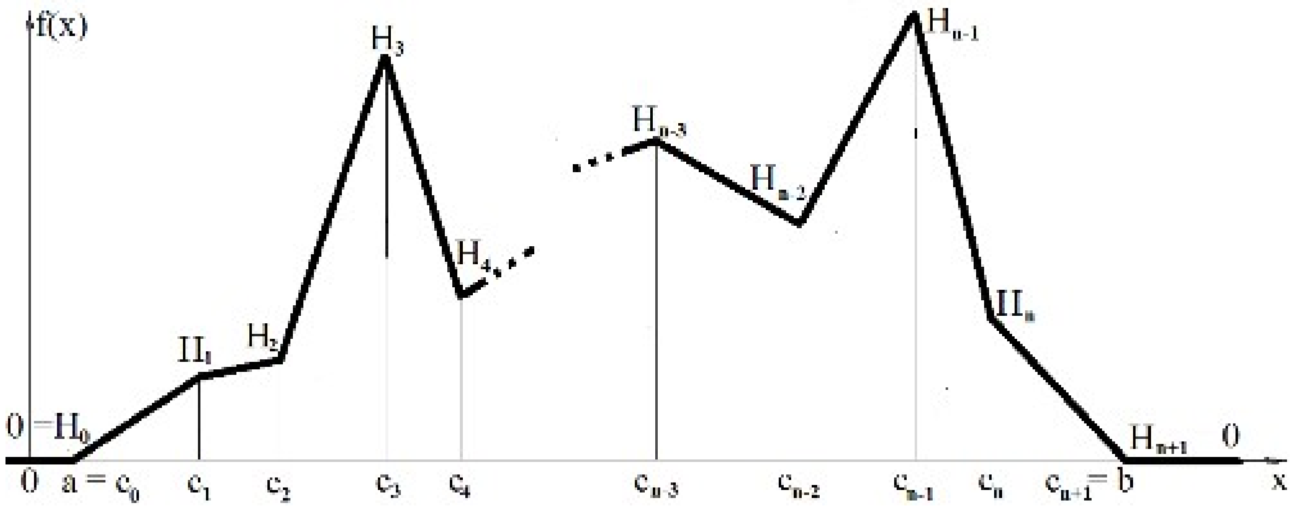

Fig. 2. General one-dimensional piecewise linear continuous probability distribution

Here probability density distribution function f(x) is as always non-negative everywhere ($-\infty < x < +\infty$) and can be positive on some finite segment (closed interval)

$$-\infty < a \leq x \leq b < +\infty \ (a < b)$$

only. Let n ($n \in N = \{1, 2, ...\}$) intermediate points c_1 , c_2 , c_3 , c_4 , … , c_{n-3} , c_{n-2} , c_{n-1} , c_n in the non-decreasing order so that

$$a \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \ldots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq b$$

divide this segment into n + 1 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a ,$$
$$c_{n+1} = b ,$$
$$c(i) = c_i \ (i = 0, 1, 2, \ldots , n + 1).$$

On each of n + 1 closed intervals

$$c_i \leq x \leq c_{i+1} \ (i = 0, 1, 2, \ldots , n),$$

probability density distribution function f(x) is linear. At n + 2 points

$$c_i \ (i = 0, 1, 2, \ldots , n + 1),$$

f(x) takes finite non-negative values

$$H_i = f(c_i),$$

respectively. Naturally, we have

$$H_0 = 0,$$
$$H_{n+1} = 0.$$

Note that

$$H_i = f(c_i) \ (i = 1, 2, \ldots , n)$$

may be any finite non-negative values. At each of n + 2 points

$$c_i \ (i = 0, 1, 2, \dots, n + 1),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are equal to one another and coincide with $f(c_i)$. Therefore, we obtain

$$H_i = L_i = R_i \ (i = 0, 1, 2, \dots, n + 1),$$

which makes it possible to apply the above formulas for a piecewise linear probability distribution to a piecewise linear continuous probability distribution.

Then on each of $n + 1$ closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1, 2, \dots, n),$$

linear probability density distribution function

$$f(x) = H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= H_i(c_{i+1} - x)/(c_{i+1} - c_i) + H_{i+1}(x - c_i)/(c_{i+1} - c_i).$$

Integral (cumulative) probability distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

is probability $P(X \le x)$ that real-number random variable X takes a real-number value not greater than x .

Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

In our case we have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$
$$= \Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} f(x)dx = \Sigma_{i=0}^{n} \int_{c(i)}^{c(i+1)} [H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^{n} \{H_i(c_{i+1} - c_i) + (H_{i+1} - H_i)[(c_{i+1}^2 - c_i^2)/2 - c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$
$$= \Sigma_{i=0}^{n} [H_i(c_{i+1} - c_i) + (H_{i+1} - H_i)(c_{i+1} - c_i)/2]$$
$$= \Sigma_{i=0}^{n} (H_i + H_{i+1})(c_{i+1} - c_i)/2$$
$$= \Sigma_{i=0}^{n} H_i(c_{i+1} - c_i)/2 + \Sigma_{i=0}^{n} H_{i+1}(c_{i+1} - c_i)/2.$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the n + 1 rectangular trapezoids, among them 2 rectangular triangles at the endpoints a and b .

Now use

$$H_0 = 0,$$
$$H_{n+1} = 0.$$

Then

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \Sigma_{i=1}^{n} H_i(c_{i+1} - c_i)/2 + \Sigma_{i=1}^{n} H_i(c_i - c_{i-1})/2$$
$$= \Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1})/2.$$

Therefore, to provide a possible (an admissible) probability density distribution function, necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1}) = 2$$

has to be satisfied.

Normalization Algorithm

Nota bene: This is one condition for

$$n + (n + 2) = 2n + 2$$

unknowns

$$H_i \ (i = 1, 2, \dots, n),$$

$$c_i \ (i = 0, 1, 2, \dots, n + 1).$$

Additionally,

$$H_i \geq 0 \ (i = 1, 2, 3, \dots, n),$$
$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \dots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} \ .$$

Generally, it is not possible to simply take any admissible values of

$$2n + 2 - 1 = 2n + 1$$

unknowns and then to determine the value of the remaining unknown via this condition because it can happen that this value is inadmissible.

A natural idea, way, and algorithm to avoid this difficulty are as follows:

1. Fix

$$c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq \dots \leq c_{n-3} \leq c_{n-2} \leq c_{n-1} \leq c_n \leq c_{n+1} \ .$$

2. Take any

$$H_i' \geq 0 \ (i = 1, 2, \dots, n)$$

so that there is at least one namely positive number among these n non-negative numbers.

3. Let

$$H_i \ (i = 1, 2, 3, \dots, n)$$

be proportional to

$$H_i' \geq 0 \ (i = 1, 2, 3, \dots, n),$$

respectively, with a common namely positive factor k so that

$$H_i = kH_i' \ (i = 1, 2, 3, \dots, n).$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1}) = 2$$

so that

$$k = 2 / \Sigma_{i=0}^n H_i'(c_{i+1} - c_{i-1}).$$

5. Explicitly determine

$$H_i = kH_i' \ (i = 1, 2, 3, \dots, n).$$

Mean Value (Mathematical Expectation)

Use the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx \ .$$

In our case we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} xf(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [H_i x + (H_{i+1} - H_i)(x^2 - c_i x)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^n \{H_i(c_{i+1}^2 - c_i^2)/2 + (H_{i+1} - H_i)[(c_{i+1}^3 - c_i^3)/3 - c_i(c_{i+1}^2 - c_i^2)/2]/(c_{i+1} - c_i)\}$$
$$= 1/6 \ \Sigma_{i=0}^n \{3H_i(c_{i+1}^2 - c_i^2) + (H_{i+1} - H_i)[2(c_{i+1}^2 + c_{i+1}c_i + c_i^2) - 3c_i(c_{i+1} + c_i)]\}$$
$$= 1/6 \ \Sigma_{i=0}^n [3H_i(c_{i+1}^2 - c_i^2) + (H_{i+1} - H_i)(2c_{i+1}^2 - c_i c_{i+1} - c_i^2)]$$
$$= 1/6 \ \Sigma_{i=0}^n (c_{i+1} - c_i)[3H_i(c_{i+1} + c_i) + (H_{i+1} - H_i)(2c_{i+1} + c_i)]$$
$$= 1/6 \ \Sigma_{i=0}^n (c_{i+1} - c_i)[H_i(c_{i+1} + 2c_i) + H_{i+1}(2c_{i+1} + c_i)]$$
$$= 1/6 \ \Sigma_{i=0}^n (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6 \ \Sigma_{i=0}^n (c_{i+1} - c_i)H_{i+1}(2c_{i+1} + c_i)$$
$$= 1/6 \ \Sigma_{i=1}^n (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6 \ \Sigma_{i=1}^n (c_i - c_{i-1})H_i(2c_i + c_{i-1})$$
$$= 1/6 \ \Sigma_{i=1}^n H_i[(c_{i+1} - c_i)(c_{i+1} + 2c_i) + (c_i - c_{i-1})(2c_i + c_{i-1})]$$
$$= 1/6 \ \Sigma_{i=1}^n H_i(c_{i+1}^2 + c_{i+1}c_i - 2c_i^2 + 2c_i^2 - c_i c_{i-1} - c_{i-1}^2)$$
$$= 1/6 \ \Sigma_{i=1}^n H_i(c_{i+1}^2 + c_{i+1}c_i - c_i c_{i-1} - c_{i-1}^2)$$
$$= 1/6 \ \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})$$

and finally

$$\mu = \Sigma_{i=1}^n H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \leq \nu) \geq 1/2$$

and

$$P(X \geq \nu) \geq 1/2.$$

For a continual real-number random variable X ,

$$P(X \leq \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \geq \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the following natural idea, way, and algorithm:

1. First consider

$$c_i \ (i = 0, 1, 2, \dots , n + 1)$$

not far from μ and determine both

$$L = \max\{i \mid \int_{-\infty}^{c(i)} f(x)dx < 1/2\}$$

and

$$R = \min\{i \mid \int_{c(i)}^{+\infty} f(x)dx < 1/2\}.$$

Then both

$$\int_{-\infty}^{c(L+1)} f(x)dx \geq 1/2$$

and

$$\int_{c(R-1)}^{+\infty} f(x)dx \geq 1/2.$$

2. On half-closed interval

$$c(L) = c_L < \nu \leq c_{L+1} = c(L+1),$$

determine

$$\nu_{min} = \inf\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}.$$

3. On half-closed interval

$$c(R-1) = c_{R-1} \leq \nu < c_R = c(R),$$

determine

$$\nu_{max} = \sup\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\}.$$

4. Then the set of all the median values ν is the interval whose endpoints are

$$\nu_{min} \leq \nu_{max}$$

each of wich is included into the interval if and only if the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and/or

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

respectively.

Notata bene:

1. If

$$\nu_{min} = \nu_{max} ,$$

then the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

hence the closed interval

$$\nu_{min} \leq \nu \leq \nu_{max}$$

contains the only median value

$$\nu = \nu_{min} = \nu_{max} .$$

2. If

$$\nu_{min} < \nu_{max} ,$$

then the integral of f(x) on the interval whose endpoints are ν_{min} and ν_{max} vanishes independently of their including or excluding. Hence on this interval, non-negative probability density distribution function f(x) also vanishes possibly excepting points whose set has zero measure (in our case, a finite set).

Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density distribution function $f(x)$ takes its maximum value f_{max} .
In particular, for a piecewise linear continuous probability distribution with probability density function $f(x)$,

$$f_{max} = \max\{f(c_i) \mid i = 1, 2, \dots , n\}.$$

If $f(x) = f_{max}$ at some x , then this x is one of the modes.
In particular, if $f(c_i) = f_{max}$ at some i , then c_i at this i is one of the modes.
Nota bene: The set of all the modes both contains separate points

$$c_i \ (i = 1, 2, \dots , n)$$

for which

$$f(c_i) = f_{sup} = f_{max}$$

and includes closed intervals

$$c_i \le x \le c_{i+1} \ (i = 1, 2, \dots , n - 1)$$

for which

$$f(c_i) = f(c_{i+1}) = f_{sup} = f_{max} .$$

Variance

Use the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

In our case we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx$$

$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} (x - \mu)^2 f(x)dx = \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} [H_i(x - \mu)^2 + (H_{i+1} - H_i)(x - \mu)^2(x - c_i)/(c_{i+1} - c_i)]dx$$

$$= \Sigma_{i=0}^n \int_{c(i)}^{c(i+1)} \{H_i(x^2 - 2\mu x + \mu^2) + (H_{i+1} - H_i)[x^3 - (2\mu + c_i)x^2 + (\mu^2 + 2\mu c_i)x - \mu^2 c_i]/(c_{i+1} - c_i)\}dx$$

$$= \Sigma_{i=0}^n \{H_i[(c_{i+1}^3 - c_i^3)/3 - 2\mu(c_{i+1}^2 - c_i^2)/2 + \mu^2(c_{i+1} - c_i)]$$

$$+ (H_{i+1} - H_i)[(c_{i+1}^4 - c_i^4)/4 - (2\mu + c_i)(c_{i+1}^3 - c_i^3)/3 + (\mu^2 + 2\mu c_i)(c_{i+1}^2 - c_i^2)/2 - \mu^2 c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$

$$= 1/12\ \Sigma_{i=0}^n \{H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)[3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - (4c_i + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) + (12\mu c_i + 6\mu^2)(c_{i+1} + c_i) - 12\mu^2 c_i]\}$$

$$= 1/12\ \Sigma_{i=0}^n [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)(3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - 4c_{i+1}^2 c_i - 4c_{i+1}c_i^2 - 4c_i^3 - 8\mu c_{i+1}^2 - 8\mu c_{i+1}c_i - 8\mu c_i^2 + 12\mu c_{i+1}c_i + 12\mu c_i^2 + 6\mu^2 c_{i+1} + 6\mu^2 c_i - 12\mu^2 c_i)]$$

$$= 1/12\ \Sigma_{i=0}^n [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$

$$= 1/12\ \Sigma_{i=0}^n [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i - 3c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 + c_i^3 + 8\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 4\mu c_i^2 - 6\mu^2 c_{i+1} + 6\mu^2 c_i)$$

$$+ H_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$

$$= 1/12\ \Sigma_{i=0}^n [H_i(c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 - 3c_i^3 + 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 2\mu^2 c_{i+1} + 2\mu^2 c_i)$$

$$+ H_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$

$$= 1/12\ \Sigma_{i=0}^n (c_{i+1} - c_i)\{H_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$

$$+ H_{i+1}[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]\}$$

$$= 1/12\ \Sigma_{i=0}^n H_i(c_{i+1} - c_i)(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$

$$+ 1/12\ \Sigma_{i=0}^n H_{i+1}(c_{i+1} - c_i)[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]$$

$$= 1/12\ \Sigma_{i=1}^n H_i(c_{i+1} - c_i)(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$

$$+ 1/12\ \Sigma_{i=1}^n H_i(c_i - c_{i-1})[3c_i^2 + 2c_i c_{i-1} + c_{i-1}^2 - 4\mu(2c_i + c_{i-1}) + 6\mu^2]$$

$$= 1/12\ \Sigma_{i=1}^{n} H_i(c_{i+1}^3 + 2c_{i+1}^2c_i + 3c_{i+1}c_i^2 + 4\mu c_{i+1}^2 - 2\mu^2c_{i+1}$$
$$- c_{i+1}^2c_i - 2c_{i+1}c_i^2 - 3c_i^3 - 4\mu c_{i+1}c_i + 2\mu^2c_i$$
$$+ 3c_i^3 + 2c_i^2c_{i-1} + c_ic_{i-1}^2 - 8\mu c_i^2 - 4\mu c_ic_{i-1} + 6\mu^2c_i$$
$$- 3c_i^2c_{i-1} - 2c_ic_{i-1}^2 - c_{i-1}^3 + 8\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 6\mu^2c_{i-1})$$
$$= 1/12\ \Sigma_{i=1}^{n} H_i(c_{i+1}^3 - c_{i-1}^3 + c_{i+1}^2c_i - c_ic_{i-1}^2 + c_{i+1}c_i^2 - c_i^2c_{i-1}$$
$$+ 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 8\mu c_i^2 + 4\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 2\mu^2c_{i+1} + 8\mu^2c_i - 6\mu^2c_{i-1})$$
$$= 1/12\ \Sigma_{i=1}^{n} H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 8\mu c_i^2 + 4\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 2\mu^2c_{i+1} + 8\mu^2c_i - 6\mu^2c_{i-1}]$$
$$= 1/12\ \Sigma_{i=1}^{n} H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu(c_{i+1}^2 - c_{i+1}c_i - 2c_i^2 + c_ic_{i-1} + c_{i-1}^2) + 2\mu^2(- c_{i+1} + 4c_i - 3c_{i-1})]$$

and finally

$$\sigma^2 = 1/12\ \Sigma_{i=1}^{n} H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu(c_{i+1}^2 - c_{i+1}c_i - 2c_i^2 + c_ic_{i-1} + c_{i-1}^2) + 2\mu^2(- c_{i+1} + 4c_i - 3c_{i-1})]$$

where

$$\mu = \Sigma_{i=1}^{n} H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

Tetragonal Probability Distribution

Main Definitions

A tetragonal probability distribution (Fig. 3) is a particular case of a general one-dimensional piecewise linear continuous probability distribution for n = 2 and further of a general one-dimensional piecewise linear probability distribution. Therefore, directly apply the above formulas for a general one-dimensional piecewise linear continuous probability distribution to a tetragonal probability distribution.

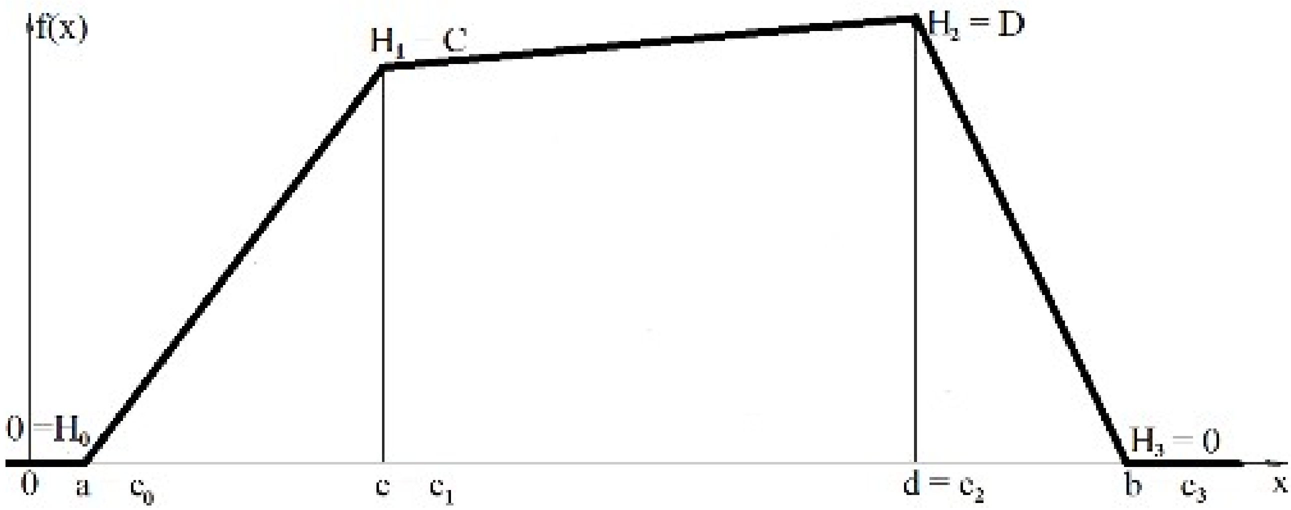

Fig. 3. Tetragonal probability distribution

Here probability density distribution function f(x) is as always non-negative everywhere ($-\infty < x <$

$+\infty$) and can be positive on some finite segment (closed interval)

$$-\infty < a \le x \le b < +\infty \ (a < b)$$

only. Let n = 2 intermediate points $c = c_1$ and $d = c_2$ in the non-decreasing order so that

$$a \le c_1 \le c_2 \le b$$

divide this segment into n + 1 = 3 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a \ ,$$
$$c_3 = b \ ,$$
$$c(i) = c_i \ (i = 0, 1, 2, 3).$$

On each of n + 1 = 3 closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1, 2),$$

probability density distribution function f(x) is linear. At n + 2 = 4 points

$$c_i \ (i = 0, 1, 2, 3),$$

f(x) takes finite non-negative values

$$H_i = f(c_i),$$

respectively. Naturally, we have

$$H_0 = 0,$$
$$H_3 = 0.$$

Note that

$$H_i = f(c_i) \ (i = 1, 2)$$

with additional natural notation

$$C = H_1 \ ,$$
$$D = H_2$$

for values f(x) at points

$$c_1 = c \ ,$$
$$c_2 = d \ ,$$

respectively, may be any finite non-negative values. At each of n + 2 = 4 points

$$c_i \ (i = 0, 1, 2, 3),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are equal to one another and coincide with $f(c_i)$. Therefore, we obtain

$$H_i = L_i = R_i \ (i = 0, 1, 2, 3),$$

which makes it possible to apply the above formulas for a piecewise linear probability distribution to a piecewise linear continuous probability distribution.

Then on each of n + 1 = 3 closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1, 2),$$

linear probability density distribution function

$$f(x) = H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= H_i(c_{i+1} - x)/(c_{i+1} - c_i) + H_{i+1}(x - c_i)/(c_{i+1} - c_i).$$

Integral (cumulative) probability distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

is probability $P(X \le x)$ that real-number random variable X takes a real-number value not greater than x .

Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

In our case we have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx$$
$$= \Sigma_{i=0}^2 \int_{c(i)}^{c(i+1)} f(x)dx = \Sigma_{i=0}^2 \int_{c(i)}^{c(i+1)} [H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^2 \{H_i(c_{i+1} - c_i) + (H_{i+1} - H_i)[(c_{i+1}^2 - c_i^2)/2 - c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$
$$= \Sigma_{i=0}^2 [H_i(c_{i+1} - c_i) + (H_{i+1} - H_i)(c_{i+1} - c_i)/2]$$
$$= \Sigma_{i=0}^2 (H_i + H_{i+1})(c_{i+1} - c_i)/2$$
$$= \Sigma_{i=0}^2 H_i(c_{i+1} - c_i)/2 + \Sigma_{i=0}^n H_{i+1}(c_{i+1} - c_i)/2.$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the $n + 1 = 3$ rectangular trapezoids, among them 2 rectangular triangles at the endpoints a and b .

Now use

$$H_0 = 0,$$
$$H_3 = 0.$$

Then

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \Sigma_{i=1}^2 H_i(c_{i+1} - c_i)/2 + \Sigma_{i=1}^2 H_i(c_i - c_{i-1})/2$$
$$= \Sigma_{i=1}^2 H_i(c_{i+1} - c_{i-1})/2.$$

Therefore, to provide a possible (an admissible) probability density distribution function, necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^2 H_i(c_{i+1} - c_{i-1}) = 2$$

has to be satisfied.

Using

$$c_0 = a ,$$
$$c_1 = c ,$$
$$c_2 = d ,$$
$$c_3 = b ,$$
$$H_1 = C ,$$
$$H_2 = D ,$$

we obtain

$$H_1(c_2 - c_0) + H_2(c_3 - c_1) = C(d - a) + D(b - c)$$

and finally

$$C(d - a) + D(b - c) = 2.$$

Normalization Algorithm

Nota bene: This is one condition for

$$n + (n + 2) = 2n + 2 = 6$$

unknowns

$$H_i \ (i = 1, 2),$$
$$c_i \ (i = 0, 1, 2, 3).$$

Additionally,

$$H_i \geq 0 \ (i = 1, 2),$$
$$c_0 \leq c_1 \leq c_2 \leq c_3 .$$

Generally, it is not possible to simply take any admissible values of

$$2n + 2 - 1 = 2n + 1 = 5$$

unknowns and then to determine the value of the remaining unknown via this condition because it can happen that this value is inadmissible.

A natural idea, way, and algorithm to avoid this difficulty are as follows:

1. Fix

$$c_0 \leq c_1 \leq c_2 \leq c_3 .$$

2. Take any

$$H_i' \geq 0 \ (i = 1, 2)$$

so that there is at least one namely positive number among these $n = 2$ non-negative numbers.

3. Let

$$H_i \ (i = 1, 2)$$

be proportional to

$$H_i' \geq 0 \ (i = 1, 2),$$

respectively, with a common namely positive factor k so that

$$H_i = kH_i' \ (i = 1, 2).$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$\Sigma_{i=1}^2 H_i(c_{i+1} - c_{i-1}) = 2$$

so that

$$k = 2 / \Sigma_{i=0}^2 H_i'(c_{i+1} - c_{i-1}).$$

5. Explicitly determine

$$H_i = kH_i' \ (i = 1, 2).$$

Using

$$c_0 = a ,$$
$$c_1 = c ,$$
$$c_2 = d ,$$
$$c_3 = b ,$$
$$H_1 = C ,$$
$$H_2 = D$$

and naturally denoting

$$H_1' = C' ,$$
$$H_2' = D' ,$$

we obtain the same algorithm in the following form:

1. Fix

$$a \leq c \leq d \leq b .$$

2. Take any

$$C' \geq 0,$$
$$D' \geq 0$$

so that there is at least one namely positive number among these n = 2 non-negative numbers.

3. Let C and D be proportional to C' and D', respectively, with a common namely positive factor k so that

$$C = kC' ,$$
$$D = kD' .$$

4. Explicitly determine the value of parameter k as the only unknown via this necessary and sufficient integral normalization condition

$$C(d - a) + D(b - c) = 2$$

so that

$$k = 2/[C(d - a) + D(b - c)].$$

5. Explicitly determine

$$C = kC' ,$$
$$D = kD' .$$

Mean Value (Mathematical Expectation)

Use the common integral definition [Cramér] of the mean value (mathematical expectation)

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx .$$

In our case we determine

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=0}^2 \int_{c(i)}^{c(i+1)} xf(x)dx = \Sigma_{i=0}^2 \int_{c(i)}^{c(i+1)} [H_i x + (H_{i+1} - H_i)(x^2 - c_i x)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^2 \{H_i(c_{i+1}^2 - c_i^2)/2 + (H_{i+1} - H_i)[(c_{i+1}^3 - c_i^3)/3 - c_i(c_{i+1}^2 - c_i^2)/2]/(c_{i+1} - c_i)\}$$

$$= 1/6\ \Sigma_{i=0}^2\ \{3H_i(c_{i+1}^2 - c_i^2) + (H_{i+1} - H_i)[2(c_{i+1}^2 + c_{i+1}c_i + c_i^2) - 3c_i(c_{i+1} + c_i)]\}$$
$$= 1/6\ \Sigma_{i=0}^2\ [3H_i(c_{i+1}^2 - c_i^2) + (H_{i+1} - H_i)(2c_{i+1}^2 - c_ic_{i+1} - c_i^2)]$$
$$= 1/6\ \Sigma_{i=0}^2\ (c_{i+1} - c_i)[3H_i(c_{i+1} + c_i) + (H_{i+1} - H_i)(2c_{i+1} + c_i)]$$
$$= 1/6\ \Sigma_{i=0}^2\ (c_{i+1} - c_i)[H_i(c_{i+1} + 2c_i) + H_{i+1}(2c_{i+1} + c_i)]$$
$$= 1/6\ \Sigma_{i=0}^2\ (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6\ \Sigma_{i=0}^2\ (c_{i+1} - c_i)H_{i+1}(2c_{i+1} + c_i)$$
$$= 1/6\ \Sigma_{i=1}^2\ (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6\ \Sigma_{i=1}^2\ (c_i - c_{i-1})H_i(2c_i + c_{i-1})$$
$$= 1/6\ \Sigma_{i=1}^2\ H_i[(c_{i+1} - c_i)(c_{i+1} + 2c_i) + (c_i - c_{i-1})(2c_i + c_{i-1})]$$
$$= 1/6\ \Sigma_{i=1}^2\ H_i(c_{i+1}^2 + c_{i+1}c_i - 2c_i^2 + 2c_i^2 - c_ic_{i-1} - c_{i-1}^2)$$
$$= 1/6\ \Sigma_{i=1}^2\ H_i(c_{i+1}^2 + c_{i+1}c_i - c_ic_{i-1} - c_{i-1}^2)$$
$$= 1/6\ \Sigma_{i=1}^2\ H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})$$

and finally

$$\mu = \Sigma_{i=1}^2\ H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Using

$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = d\ ,$$
$$c_3 = b\ ,$$
$$H_1 = C\ ,$$
$$H_2 = D\ ,$$

we obtain the same formula in the following form:

$$\mu = [H_1(c_2 - c_0)(c_2 + c_1 + c_0) + H_2(c_3 - c_1)(c_3 + c_2 + c_1)]/6,$$
$$\mu = [C(d - a)(d + c + a) + D(b - c)(b + d + c)]/6,$$

and finally

$$\mu = [C(d - a)(a + c + d) + D(b - c)(b + c + d)]/6.$$

Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \le \nu) \ge 1/2$$

and

$$P(X \ge \nu) \ge 1/2.$$

For a continual real-number random variable X ,

$$P(X \le \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \ge \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the following natural idea, way, and algorithm:

1. First consider

$$c_i\ (i = 0, 1, 2, \dots , n + 1)$$

not far from μ and determine both

$$L = \max\{i \mid \int_{-\infty}^{c(i)} f(x)dx < 1/2\}$$

and

$$R = \min\{i \mid \int_{c(i)}^{+\infty} f(x)dx < 1/2\}.$$

Then both

$$\int_{-\infty}^{c(L+1)} f(x)dx \ge 1/2$$

and

$$\int_{c(R-1)}^{+\infty} f(x)dx \ge 1/2.$$

2. On half-closed interval

$$c(L) = c_L < \nu \le c_{L+1} = c(L+1),$$

determine

$$\nu_{min} = \inf\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}.$$

3. On half-closed interval

$$c(R-1) = c_{R-1} \le \nu < c_R = c(R),$$

determine

$$\nu_{max} = \sup\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\}.$$

4. Then the set of all the median values ν is the interval whose endpoints are

$$\nu_{min} \leq \nu_{max}$$

each of wich is included into the interval if and only if the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and/or

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

respectively.

Notata bene:

1. If

$$\nu_{min} = \nu_{max} ,$$

then the corresponding greatest lower and/or least upper bound is really taken so that

$$\nu_{min} = \min\{\nu \mid \int_{-\infty}^{\nu} f(x)dx = 1/2\}$$

and

$$\nu_{max} = \max\{\nu \mid \int_{\nu}^{+\infty} f(x)dx = 1/2\},$$

hence the closed interval

$$\nu_{min} \leq \nu \leq \nu_{max}$$

contains the only median value

$$\nu = \nu_{min} = \nu_{max} .$$

2. If

$$\nu_{min} < \nu_{max} ,$$

then the integral of f(x) on the interval whose endpoints are ν_{min} and ν_{max} vanishes independently of their including or excluding. Hence on this interval, non-negative probability density distribution function f(x) also vanishes possibly excepting points whose set has zero measure (in our case, a finite set).

Using n = 2, make the same natural idea, way, and algorithm much more explicit:

1. First determine both

$$F(c) = \int_{-\infty}^{c} f(x)dx = \int_{a}^{c} f(x)dx = \int_{a}^{c} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{c}(x - a)dx = C/(c - a)\, [(c^2 - a^2)/2 - a(c - a)]$$
$$= C[(c + a)/2 - a] = C(c - a)/2$$

and

$$F(d) = 1 - \int_{d}^{+\infty} f(x)dx = 1 - \int_{d}^{b} f(x)dx = 1 - \int_{d}^{b} D(b - x)/(b - d)\, dx$$
$$= 1 - D/(b - d) \int_{d}^{b}(b - x)dx = 1 - D/(b - d)\, [b(b - d) - (b^2 - d^2)/2]$$
$$= 1 - D[b - (b + d)/2] = 1 - D(b - d)/2.$$

2. If

$$F(c) > 1/2,$$

or, equivalently,

$$C(c - a) > 1,$$

then there is the only median value ν strictly between a and c so that

$$F(\nu) = 1/2,$$
$$F(\nu) = \int_{-\infty}^{\nu} f(x)dx = \int_{a}^{\nu} f(x)dx = \int_{a}^{\nu} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{\nu}(x - a)dx = C/(c - a)\, [(\nu^2 - a^2)/2 - a(\nu - a)]$$
$$= C/(c - a)\, (\nu - a)^2/2 = 1/2,$$
$$(\nu - a)^2 = (c - a)/C ,$$
$$\nu = a + [(c - a)/C]^{1/2} .$$

3. If

$$F(c) = 1/2,$$

or, equivalently,

$$C(c - a) = 1,$$

then there is the only median value

$$\nu = c .$$

4. If

$$F(d) < 1/2,$$

or, equivalently,

$$1 - D(b - d)/2 < 1/2,$$
$$D(b - d) > 1,$$

then there is the only median value ν strictly between d and b so that

$$F(\nu) = 1/2,$$
$$F(\nu) = 1 - \int_\nu^{+\infty} f(x)dx = 1 - \int_\nu^b f(x)dx = 1 - \int_\nu^b D(b - x)/(b - d)\, dx$$
$$= 1 - D/(b - d) \int_\nu^b (b - x)dx = 1 - D/(b - d)\, [b(b - \nu) - (b^2 - \nu^2)/2]$$
$$= 1 - D/(b - d)\, (b - \nu)^2/2 = 1/2,$$
$$D/(b - d)\, (b - \nu)^2 = 1,$$
$$(b - \nu)^2 = (b - d)/D ,$$
$$\nu = b - [(b - d)/D]^{1/2} .$$

5. If

$$F(d) = 1/2,$$

or, equivalently,

$$1 - D(b - d)/2 = 1/2,$$
$$D(b - d) = 1,$$

then there is the only median value

$$\nu = d .$$

6. Finally, if

$$F(c) < 1/2 < F(d),$$

or, equivalently,

$$C(c - a) < 1$$

and

$$D(b - d) < 1,$$

then there is the only median value ν strictly between c and d ($c < \nu < d$) because incremental distribution function F(c) strictly monotonically increases on this interval (c , d) so that

$$F(\nu) = 1/2,$$
$$F(\nu) = \int_{-\infty}^\nu f(x)dx = \int_a^\nu f(x)dx = \int_a^c f(x)dx + \int_c^\nu f(x)dx$$
$$= F(c) + \int_c^\nu [C(d - x) + D(x - c)]/(d - c)\, dx$$
$$= C(c - a)/2 + \{C[d(\nu - c) - (\nu^2 - c^2)/2] + D[(\nu^2 - c^2)/2 - c(\nu - c)]\}/(d - c)$$
$$= C(c - a)/2 + [(Cd - Dc)(\nu - c) + (D - C)(\nu^2 - c^2)/2]/(d - c) = 1/2,$$
$$C(c - a)(d - c) + 2(Cd - Dc)(\nu - c) + (D - C)(\nu^2 - c^2) = d - c ,$$
$$(D - C)\nu^2 + 2(Cd - Dc)\nu + C(c - a)(d - c) - 2(Cd - Dc)c - (D - C)c^2 + c - d = 0.$$

6.1. If D = C and naturally positive, then

$$2C(d - c)\nu + C(c - a)(d - c) - 2C(d - c)c + c - d = 0,$$
$$2C\nu = 1 + C(a + c),$$
$$\nu = 1/(2C) + (a + c)/2.$$

Directly moving from left to right, we also obtain the same result

$$\nu = c + [1/2 - C(c - a)/2]/C$$

at once. We have

$$\nu - c = 1/(2C) + (a - c)/2 > 0$$

because

$$C(c - a) < 1.$$

Directly moving from right to left, we obtain

$$\nu = d - [1/2 - C(b - d)/2]/C = - 1/(2C) + d + (b - d)/2 = (b + d)/2 - 1/(2C)$$

at once. We have

$$d - \nu = d + 1/(2C) - (b + d)/2 > 0$$

because

$$C(b - d) < 1.$$

To prove the equivalence of these both formulas

$$\nu = 1/(2C) + (a + c)/2$$

and

$$\nu = (b + d)/2 - 1/(2C)$$

for ν , note that

$$1/(2C) + (a + c)/2 = (b + d)/2 - 1/(2C)$$

because the normalization condition

$$C(c - a)/2 + C(d - c) + C(b - d)/2 = 1$$

gives

$$(b - a + d - c)/2 = 1/C .$$

6.2. If $D \neq C$, then there is the only median value ν strictly between c and d ($c < \nu < d$) because incremental distribution function F(c) strictly monotonically increases on this interval (c , d) so that

$$F(\nu) = 1/2.$$

Hence quadratic equation

$$(D - C)\nu^2 + 2(Cd - Dc)\nu + C(c - a)(d - c) - 2(Cd - Dc)c - (D - C)c^2 + c - d = 0$$

in ν has exactly one solution on this interval (c , d).

Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density distribution function f(x) takes its maximum value f_{max} .
If C = D and naturally positive, then there are two modes c and d .
If C > D , then there is the only mode c .
If C < D , then there is the only mode d .

Variance

Use the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

In our case n = 2 we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx$$
$$= \Sigma_{i=0}^2 \int_{c(i)}^{c(i+1)} (x - \mu)^2 f(x)dx = \Sigma_{i=0}^2 \int_{c(i)}^{c(i+1)} [H_i(x - \mu)^2 + (H_{i+1} - H_i)(x - \mu)^2(x - c_i)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^2 \int_{c(i)}^{c(i+1)} \{H_i(x^2 - 2\mu x + \mu^2) + (H_{i+1} - H_i)[x^3 - (2\mu + c_i)x^2 + (\mu^2 + 2\mu c_i)x - \mu^2 c_i]/(c_{i+1} - c_i)\}dx$$
$$= \Sigma_{i=0}^2 \{H_i[(c_{i+1}^3 - c_i^3)/3 - 2\mu(c_{i+1}^2 - c_i^2)/2 + \mu^2(c_{i+1} - c_i)]$$
$$+ (H_{i+1} - H_i)[(c_{i+1}^4 - c_i^4)/4 - (2\mu + c_i)(c_{i+1}^3 - c_i^3)/3 + (\mu^2 + 2\mu c_i)(c_{i+1}^2 - c_i^2)/2 - \mu^2 c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$
$$= 1/12\ \Sigma_{i=0}^2 \{H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)[3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - (4c_i + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) + (12\mu c_i + 6\mu^2)(c_{i+1} + c_i) - 12\mu^2 c_i]\}$$
$$= 1/12\ \Sigma_{i=0}^2 [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)(3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 + 3c_i^3 - 4c_{i+1}^2 c_i - 4c_{i+1}c_i^2 - 4c_i^3 - 8\mu c_{i+1}^2 - 8\mu c_{i+1}c_i - 8\mu c_i^2 + 12\mu c_{i+1}c_i + 12\mu c_i^2 + 6\mu^2 c_{i+1} + 6\mu^2 c_i - 12\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^2 [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^2 [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i - 3c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 + c_i^3 + 8\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 4\mu c_i^2 - 6\mu^2 c_{i+1} + 6\mu^2 c_i)$$
$$+ H_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^2 [H_i(c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 - 3c_i^3 + 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 2\mu^2 c_{i+1} + 2\mu^2 c_i)$$
$$+ H_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^2 (c_{i+1} - c_i)\{H_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$

$$+ H_{i+1}[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]\}$$
$$= 1/12\ \Sigma_{i=0}^2\ H_i(c_{i+1} - c_i)(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$
$$+ 1/12\ \Sigma_{i=0}^2\ H_{i+1}(c_{i+1} - c_i)[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]$$
$$= 1/12\ \Sigma_{i=1}^2\ H_i(c_{i+1} - c_i)(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$
$$+ 1/12\ \Sigma_{i=1}^2\ H_i(c_i - c_{i-1})[3c_i^2 + 2c_ic_{i-1} + c_{i-1}^2 - 4\mu(2c_i + c_{i-1}) + 6\mu^2]$$
$$= 1/12\ \Sigma_{i=1}^2\ H_i(c_{i+1}^3 + 2c_{i+1}^2c_i + 3c_{i+1}c_i^2 + 4\mu c_{i+1}^2 - 2\mu^2c_{i+1}$$
$$- c_{i+1}^2c_i - 2c_{i+1}c_i^2 - 3c_i^3 - 4\mu c_{i+1}c_i + 2\mu^2c_i$$
$$+ 3c_i^3 + 2c_i^2c_{i-1} + c_ic_{i-1}^2 - 8\mu c_i^2 - 4\mu c_ic_{i-1} + 6\mu^2c_i$$
$$- 3c_i^2c_{i-1} - 2c_ic_{i-1}^2 - c_{i-1}^3 + 8\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 6\mu^2c_{i-1})$$
$$= 1/12\ \Sigma_{i=1}^2\ H_i(c_{i+1}^3 - c_{i-1}^3 + c_{i+1}^2c_i - c_ic_{i-1}^2 + c_{i+1}c_i^2 - c_i^2c_{i-1}$$
$$+ 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 8\mu c_i^2 + 4\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 2\mu^2c_{i+1} + 8\mu^2c_i - 6\mu^2c_{i-1})$$
$$= 1/12\ \Sigma_{i=1}^2\ H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 8\mu c_i^2 + 4\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 2\mu^2c_{i+1} + 8\mu^2c_i - 6\mu^2c_{i-1}]$$
$$= 1/12\ \Sigma_{i=1}^2\ H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu(c_{i+1}^2 - c_{i+1}c_i - 2c_i^2 + c_ic_{i-1} + c_{i-1}^2) + 2\mu^2(- c_{i+1} + 4c_i - 3c_{i-1})]$$

and finally

$$\sigma^2 = 1/12\ \Sigma_{i=1}^2\ H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu(c_{i+1}^2 - c_{i+1}c_i - 2c_i^2 + c_ic_{i-1} + c_{i-1}^2) + 2\mu^2(- c_{i+1} + 4c_i - 3c_{i-1})]$$

where

$$\mu = \Sigma_{i=1}^2\ H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Using

$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = d\ ,$$
$$c_3 = b\ ,$$
$$H_1 = C\ ,$$
$$H_2 = D\ ,$$

we obtain the same formulas in the following forms:

$$\mu = [H_1(c_2 - c_0)(c_2 + c_1 + c_0) + H_2(c_3 - c_1)(c_3 + c_2 + c_1)]/6,$$
$$\mu = [C(d - a)(a + c + d) + D(b - c)(b + c + d)]/6,$$

as well as

$$\sigma^2 = 1/12\ \{H_1[(c_2 - c_0)(c_2^2 + c_2c_1 + c_2c_0 + c_1^2 + c_1c_0 + c_0^2)$$
$$+ 4\mu(c_2^2 - c_2c_1 - 2c_1^2 + c_1c_0 + c_0^2) + 2\mu^2(- c_2 + 4c_1 - 3c_0)]$$
$$+ H_2[(c_3 - c_1)(c_3^2 + c_3c_2 + c_3c_1 + c_2^2 + c_2c_1 + c_1^2)$$
$$+ 4\mu(c_3^2 - c_3c_2 - 2c_2^2 + c_2c_1 + c_1^2) + 2\mu^2(- c_3 + 4c_2 - 3c_1)]\}$$

and hence

$$\sigma^2 = \{C[(d - a)(d^2 + dc + da + c^2 + ca + a^2)$$
$$+ 4\mu(d^2 - dc - 2c^2 + ca + a^2) + 2\mu^2(- d + 4c - 3a)]$$
$$+ D[(b - c)(b^2 + bd + bc + d^2 + dc + c^2)$$
$$+ 4\mu(b^2 - bd - 2d^2 + dc + c^2) + 2\mu^2(- b + 4d - 3c)]\}/12.$$

Finally

$$\sigma^2 = \{C[(d - a)(a^2 + ac + ad + c^2 + cd + d^2)$$
$$+ 4\mu(a^2 + ac - 2c^2 - cd + d^2) + 2\mu^2(- 3a + 4c - d)]$$
$$+ D[(b - c)(b^2 + bc + bd + c^2 + cd + d^2)$$
$$+ 4\mu(b^2 - bd + c^2 + cd - 2d^2) + 2\mu^2(- b - 3c + 4d)]\}/12.$$

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

Piecewise Linear Probability Distribution Formulas Verification via Triangular Probability Distribution

Main Definitions

Verify formulas for a general one-dimensional piecewise linear probability distribution using formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution as a particular case of a general one-dimensional piecewise linear continuous probability distribution for n = 1 and further of a general one-dimensional piecewise linear probability distribution. Therefore, directly apply the above formulas for a general one-dimensional piecewise linear continuous probability distribution and namely for a tetragonal probability distribution to a triangular probability distribution (Fig. 4).

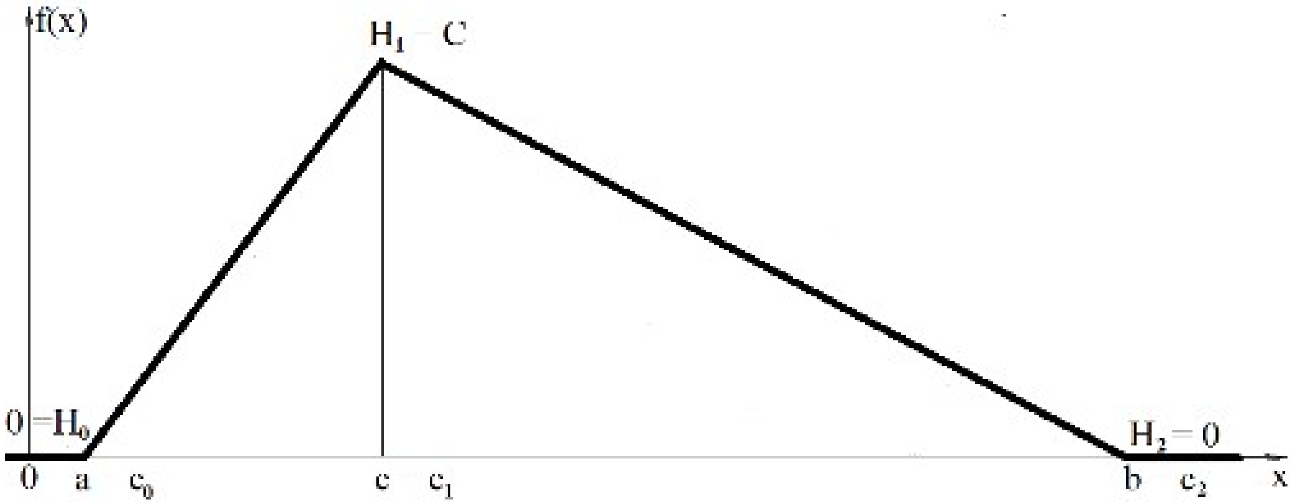

Fig. 4. Triangular probability distribution

Here probability density distribution function f(x) is as always non-negative everywhere ($-\infty < x < +\infty$) and can be positive on some finite segment (closed interval)

$$-\infty < a \le x \le b < +\infty \ (a < b)$$

only. Let n = 1 intermediate point $c = c_1$ so that

$$a \le c_1 \le b$$

divide this segment into n + 1 = 2 parts (pieces) of generally different lengths. To unify the notation, denote

$$c_0 = a \ ,$$

$$c_2 = b \ ,$$

$$c(i) = c_i \ (i = 0, 1, 2).$$

On each of n + 1 = 2 closed intervals

$$c_i \le x \le c_{i+1} \ (i = 0, 1),$$

probability density distribution function f(x) is linear. At n + 2 = 3 points

$$c_i \ (i = 0, 1, 2),$$

f(x) takes finite non-negative values

$$H_i = f(c_i),$$

respectively. Naturally, we have

$$H_0 = 0,$$
$$H_2 = 0.$$

Note that

$$H_1 = f(c_1)$$

with additional natural notation

$$C = H_1$$

for value f(x) at point

$$c_1 = c$$

may be any finite positive value. At each of n + 2 = 3 points

$$c_i \ (i = 0, 1, 2),$$

left and right one-sided limits

$$\lim f(x) = L_i \ (x \to c_i - 0),$$
$$\lim f(x) = R_i \ (x \to c_i + 0)$$

are equal to one another and coincide with $f(c_i)$. Therefore, we obtain

$$H_i = L_i = R_i \ (i = 0, 1, 2),$$

which makes it possible to apply the above formulas for a piecewise linear probability distribution to a piecewise linear continuous probability distribution.
Then on each of n + 1 = 2 closed intervals

$$c_i \leq x \leq c_{i+1} \ (i = 0, 1),$$

linear probability density distribution function

$$f(x) = H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)$$
$$= H_i(c_{i+1} - x)/(c_{i+1} - c_i) + H_{i+1}(x - c_i)/(c_{i+1} - c_i).$$

Integral (cumulative) probability distribution function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

is probability $P(X \leq x)$ that real-number random variable X takes a real-number value not greater than x .

Normalization Condition

The probability of the event that X takes any finite real value is namely 1 because this event is certain. This gives integral normalization condition

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

In our case we have

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \int_{a}^{b} f(x)dx$$
$$= \Sigma_{i=0}^{1} \int_{c(i)}^{c(i+1)} f(x)dx = \Sigma_{i=0}^{1} \int_{c(i)}^{c(i+1)} [H_i + (H_{i+1} - H_i)(x - c_i)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^{1} \{H_i(c_{i+1} - c_i) + (H_{i+1} - H_i)[(c_{i+1}^2 - c_i^2)/2 - c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$
$$= \Sigma_{i=0}^{1} [H_i(c_{i+1} - c_i) + (H_{i+1} - H_i)(c_{i+1} - c_i)/2]$$
$$= \Sigma_{i=0}^{1} (H_i + H_{i+1})(c_{i+1} - c_i)/2$$
$$= \Sigma_{i=0}^{1} H_i(c_{i+1} - c_i)/2 + \Sigma_{i=0}^{n} H_{i+1}(c_{i+1} - c_i)/2.$$

We can also obtain this result at once rather geometrically than analytically, namely via adding the areas of the 2 rectangular triangles.
Now use

$$H_0 = 0,$$
$$H_3 = 0.$$

Then

$$1 = \int_{-\infty}^{+\infty} f(x)dx = \Sigma_{i=1}^{1} H_i(c_{i+1} - c_i)/2 + \Sigma_{i=1}^{2} H_i(c_i - c_{i-1})/2$$
$$= \Sigma_{i=1}^{1} H_i(c_{i+1} - c_{i-1})/2 = H_1(c_2 - c_0)/2.$$

Therefore, to provide a possible (an admissible) probability density distribution function, necessary and sufficient integral normalization condition

$$H_1(c_2 - c_0) = 2$$
has to be satisfied.
Using
$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = b\ ,$$
$$H_1 = C\ ,$$
we obtain
$$H_1(c_2 - c_0) = C(b - a)$$
and finally
$$C(b - a) = 2,$$
$$C = 2/(b - a).$$
The known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

Mean Value (Mathematical Expectation)

Use the common integral definition [Cramér] of the mean value (mathematical expectation)
$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx\ .$$
In our case we determine
$$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_a^b xf(x)dx$$
$$= \Sigma_{i=0}^1 \int_{c(i)}^{c(i+1)} xf(x)dx = \Sigma_{i=0}^1 \int_{c(i)}^{c(i+1)} [H_ix + (H_{i+1} - H_i)(x^2 - c_ix)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^1 \{H_i(c_{i+1}^2 - c_i^2)/2 + (H_{i+1} - H_i)[(c_{i+1}^3 - c_i^3)/3 - c_i(c_{i+1}^2 - c_i^2)/2]/(c_{i+1} - c_i)\}$$
$$= 1/6\ \Sigma_{i=0}^1 \{3H_i(c_{i+1}^2 - c_i^2) + (H_{i+1} - H_i)[2(c_{i+1}^2 + c_{i+1}c_i + c_i^2) - 3c_i(c_{i+1} + c_i)]\}$$
$$= 1/6\ \Sigma_{i=0}^1 [3H_i(c_{i+1}^2 - c_i^2) + (H_{i+1} - H_i)(2c_{i+1}^2 - c_ic_{i+1} - c_i^2)]$$
$$= 1/6\ \Sigma_{i=0}^1 (c_{i+1} - c_i)[3H_i(c_{i+1} + c_i) + (H_{i+1} - H_i)(2c_{i+1} + c_i)]$$
$$= 1/6\ \Sigma_{i=0}^1 (c_{i+1} - c_i)[H_i(c_{i+1} + 2c_i) + H_{i+1}(2c_{i+1} + c_i)]$$
$$= 1/6\ \Sigma_{i=0}^1 (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6\ \Sigma_{i=0}^2 (c_{i+1} - c_i)H_{i+1}(2c_{i+1} + c_i)$$
$$= 1/6\ \Sigma_{i=0}^1 (c_{i+1} - c_i)H_i(c_{i+1} + 2c_i) + 1/6\ \Sigma_{i=1}^2 (c_i - c_{i-1})H_i(2c_i + c_{i-1})$$
$$= 1/6\ \Sigma_{i=0}^1 H_i[(c_{i+1} - c_i)(c_{i+1} + 2c_i) + (c_i - c_{i-1})(2c_i + c_{i-1})]$$
$$= 1/6\ \Sigma_{i=0}^1 H_i(c_{i+1}^2 + c_{i+1}c_i - 2c_i^2 + 2c_i^2 - c_ic_{i-1} - c_{i-1}^2)$$
$$= 1/6\ \Sigma_{i=1}^1 H_i(c_{i+1}^2 + c_{i+1}c_i - c_ic_{i-1} - c_{i-1}^2)$$
$$= 1/6\ \Sigma_{i=1}^1 H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})$$
and finally
$$\mu = H_1(c_2 - c_0)(c_2 + c_1 + c_0)/6.$$
Using
$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = b\ ,$$
$$H_1 = C\ ,$$
$$H_2 = D\ ,$$
we obtain the same formula in the following form:
$$\mu = H_1(c_2 - c_0)(c_2 + c_1 + c_0)/6,$$
$$\mu = C(b - a)(b + c + a)/6.$$
Using
$$C = 2/(b - a),$$
finally obtain
$$\mu = (a + b + c)/3.$$
The known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

Median Values

Use the common integral definition [Cramér] of median values ν for any of which both

$$P(X \le \nu) \ge 1/2$$

and

$$P(X \ge \nu) \ge 1/2.$$

For a continual real-number random variable X ,

$$P(X \le \nu) = \int_{-\infty}^{\nu} f(x)dx = P(X \ge \nu) = \int_{\nu}^{+\infty} f(x)dx = 1/2.$$

To determine the set of all the median values ν , we can use the above natural idea, way, and algorithm. Using n = 1 and the above formulas for a tetragonal probability distribution with

$$d = c ,$$
$$D = C ,$$
$$C = 2/(b - a),$$

make the same natural idea, way, and algorithm much more explicit:

1. First determine

$$F(c) = \int_{-\infty}^{c} f(x)dx = \int_{a}^{c} f(x)dx = \int_{a}^{c} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{c}(x - a)dx = C/(c - a)\, [(c^2 - a^2)/2 - a(c - a)]$$
$$= C[(c + a)/2 - a] = C(c - a)/2 = (c - a)/(b - a).$$

2. If

$$F(c) > 1/2,$$

or, equivalently,

$$c > (a + b)/2,$$

then there is the only median value ν strictly between a and c so that

$$F(\nu) = 1/2,$$
$$F(\nu) = \int_{-\infty}^{\nu} f(x)dx = \int_{a}^{\nu} f(x)dx = \int_{a}^{\nu} C(x - a)/(c - a)\, dx$$
$$= C/(c - a) \int_{a}^{\nu}(x - a)dx = C/(c - a)\, [(\nu^2 - a^2)/2 - a(\nu - a)]$$
$$= C/(c - a)\, (\nu - a)^2/2 = 1/2,$$
$$(\nu - a)^2 = (c - a)/C ,$$
$$\nu = a + [(c - a)/C]^{1/2} ,$$
$$\nu = a + [(b - a)(c - a)/2]^{1/2} .$$

The known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

3. If

$$F(c) = 1/2,$$

or, equivalently,

$$c = (a + b)/2,$$

then there is the only median value

$$\nu = c = (a + b)/2.$$

Naturally, the known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same obvious result.

4. If

$$F(c) < 1/2,$$

or, equivalently,

$$c < (a + b)/2,$$

then there is the only median value ν strictly between c and b so that

$$F(\nu) = 1/2,$$
$$F(\nu) = 1 - \int_{\nu}^{+\infty} f(x)dx = 1 - \int_{\nu}^{b} f(x)dx = 1 - \int_{\nu}^{b} C(b - x)/(b - c)\, dx$$
$$= 1 - C/(b - c) \int_{\nu}^{b}(b - x)dx = 1 - C/(b - c)\, [b(b - \nu) - (b^2 - \nu^2)/2]$$
$$= 1 - C/(b - c)\, (b - \nu)^2/2 = 1/2,$$
$$C/(b - c)\, (b - \nu)^2 = 1,$$

$$(b - \nu)^2 = (b - c)/C ,$$
$$\nu = b - [(b - c)/C]^{1/2}$$
$$\nu = b - [(b - a)(b - c)/2]^{1/2} .$$

The known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same result.

These three conditional formulas for the only median value ν can be unified as follows:

$$\nu = (a + b)/2 + \{[(b - a)(b - a + |2c - a - b|)]^{1/2} + a - b\}/2 \text{ sign}(2c - a - b).$$

In fact, we obtain:

1) by $c > (a + b)/2$,

$$\nu = (a + b)/2 + \{[(b - a)(b - a + 2c - a - b)]^{1/2} + a - b\}/2$$
$$= (a + b)/2 + \{[(b - a)(2c - 2a)]^{1/2} + a - b\}/2$$
$$= a + [(b - a)(c - a)/2]^{1/2} ;$$

2) by $c = (a + b)/2$,

$$\nu = (a + b)/2;$$

3) by $c < (a + b)/2$,

$$\nu = (a + b)/2 - \{[(b - a)(b - a - 2c + a + b)]^{1/2} + a - b\}/2$$
$$= (a + b)/2 - \{[(b - a)(2b - 2c)]^{1/2} + a - b\}/2$$
$$= b - [(b - a)(b - c)/2]^{1/2} .$$

Mode Values

To begin with, consider the common definition [Cramér] of mode values for any of which probability density distribution function $f(x)$ takes its maximum value f_{max} .

In our case, there is the only mode c .

Naturally, the known formulas [Cramér, Kotz Dorp, Wikipedia Triangular distribution] for a triangular probability distribution give the same obvious result.

Variance

Use the common integral definition [Cramér] of the variance σ^2 of a random variable X as its second central moment, namely the squared standard deviation σ , or the expected value of the squared deviation from the mean:

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx .$$

In our case n = 1 we determine

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx = \int_a^b (x - \mu)^2 f(x)dx$$
$$= \Sigma_{i=0}^1 \int_{c(i)}^{c(i+1)} (x - \mu)^2 f(x)dx = \Sigma_{i=0}^1 \int_{c(i)}^{c(i+1)} [H_i(x - \mu)^2 + (H_{i+1} - H_i)(x - \mu)^2(x - c_i)/(c_{i+1} - c_i)]dx$$
$$= \Sigma_{i=0}^1 \int_{c(i)}^{c(i+1)} \{H_i(x^2 - 2\mu x + \mu^2) + (H_{i+1} - H_i)[x^3 - (2\mu + c_i)x^2 + (\mu^2 + 2\mu c_i)x - \mu^2 c_i]/(c_{i+1} - c_i)\}dx$$
$$= \Sigma_{i=0}^1 \{H_i[(c_{i+1}^3 - c_i^3)/3 - 2\mu(c_{i+1}^2 - c_i^2)/2 + \mu^2(c_{i+1} - c_i)]$$
$$+ (H_{i+1} - H_i)[(c_{i+1}^4 - c_i^4)/4 - (2\mu + c_i)(c_{i+1}^3 - c_i^3)/3 + (\mu^2 + 2\mu c_i)(c_{i+1}^2 - c_i^2)/2 - \mu^2 c_i(c_{i+1} - c_i)]/(c_{i+1} - c_i)\}$$
$$= 1/12\ \Sigma_{i=0}^1 \{H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)[3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 +$$
$$3c_i^3 - (4c_i + 8\mu)(c_{i+1}^2 + c_{i+1}c_i + c_i^2) + (12\mu c_i + 6\mu^2)(c_{i+1} + c_i) - 12\mu^2 c_i]\}$$
$$= 1/12\ \Sigma_{i=0}^1 [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)(3c_{i+1}^3 + 3c_{i+1}^2 c_i + 3c_{i+1}c_i^2 +$$
$$3c_i^3 - 4c_{i+1}^2 c_i - 4c_{i+1}c_i^2 - 4c_i^3 - 8\mu c_{i+1}^2 - 8\mu c_{i+1}c_i - 8\mu c_i^2 + 12\mu c_{i+1}c_i + 12\mu c_i^2 + 6\mu^2 c_{i+1} + 6\mu^2 c_i - 12\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^1 [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i) + (H_{i+1} - H_i)(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 -$$
$$8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^1 [H_i(4c_{i+1}^3 - 4c_i^3 - 4\mu c_{i+1}^2 + 4\mu c_i^2 + 4\mu^2 c_{i+1} - 4\mu^2 c_i - 3c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 + c_i^3 + 8\mu c_{i+1}^2 -$$
$$4\mu c_{i+1}c_i - 4\mu c_i^2 - 6\mu^2 c_{i+1} + 6\mu^2 c_i)$$
$$+ H_{i+1}(3c_{i+1}^3 - c_{i+1}^2 c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2 c_{i+1} - 6\mu^2 c_i)]$$
$$= 1/12\ \Sigma_{i=0}^1 [H_i(c_{i+1}^3 + c_{i+1}^2 c_i + c_{i+1}c_i^2 - 3c_i^3 + 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 2\mu^2 c_{i+1} + 2\mu^2 c_i)$$

$$+ H_{i+1}(3c_{i+1}^3 - c_{i+1}^2c_i - c_{i+1}c_i^2 - c_i^3 - 8\mu c_{i+1}^2 + 4\mu c_{i+1}c_i + 4\mu c_i^2 + 6\mu^2c_{i+1} - 6\mu^2c_i)]$$
$$= 1/12\ \Sigma_{i=0}^1\ (c_{i+1} - c_i)\{H_i(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$
$$+ H_{i+1}[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]\}$$
$$= 1/12\ \Sigma_{i=0}^1\ H_i(c_{i+1} - c_i)(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$
$$+ 1/12\ \Sigma_{i=0}^1\ H_{i+1}(c_{i+1} - c_i)[3c_{i+1}^2 + 2c_{i+1}c_i + c_i^2 - 4\mu(2c_{i+1} + c_i) + 6\mu^2]$$
$$= 1/12\ \Sigma_{i=1}^1\ H_i(c_{i+1} - c_i)(c_{i+1}^2 + 2c_{i+1}c_i + 3c_i^2 + 4\mu c_{i+1} - 2\mu^2)$$
$$+ 1/12\ \Sigma_{i=1}^1\ H_i(c_i - c_{i-1})[3c_i^2 + 2c_ic_{i-1} + c_{i-1}^2 - 4\mu(2c_i + c_{i-1}) + 6\mu^2]$$
$$= 1/12\ \Sigma_{i=1}^1\ H_i(c_{i+1}^3 + 2c_{i+1}^2c_i + 3c_{i+1}c_i^2 + 4\mu c_{i+1}^2 - 2\mu^2c_{i+1}$$
$$- c_{i+1}^2c_i - 2c_{i+1}c_i^2 - 3c_i^3 - 4\mu c_{i+1}c_i + 2\mu^2c_i$$
$$+ 3c_i^3 + 2c_i^2c_{i-1} + c_ic_{i-1}^2 - 8\mu c_i^2 - 4\mu c_ic_{i-1} + 6\mu^2c_i$$
$$- 3c_i^2c_{i-1} - 2c_ic_{i-1}^2 - c_{i-1}^3 + 8\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 6\mu^2c_{i-1})$$
$$= 1/12\ \Sigma_{i=1}^1\ H_i(c_{i+1}^3 - c_{i-1}^3 + c_{i+1}^2c_i - c_ic_{i-1}^2 + c_{i+1}c_i^2 - c_i^2c_{i-1}$$
$$+ 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 8\mu c_i^2 + 4\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 2\mu^2c_{i+1} + 8\mu^2c_i - 6\mu^2c_{i-1})$$
$$= 1/12\ \Sigma_{i=1}^1\ H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu c_{i+1}^2 - 4\mu c_{i+1}c_i - 8\mu c_i^2 + 4\mu c_ic_{i-1} + 4\mu c_{i-1}^2 - 2\mu^2c_{i+1} + 8\mu^2c_i - 6\mu^2c_{i-1}]$$
$$= 1/12\ \Sigma_{i=1}^1\ H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu(c_{i+1}^2 - c_{i+1}c_i - 2c_i^2 + c_ic_{i-1} + c_{i-1}^2) + 2\mu^2(- c_{i+1} + 4c_i - 3c_{i-1})]$$

and finally

$$\sigma^2 = 1/12\ \Sigma_{i=1}^1\ H_i[(c_{i+1} - c_{i-1})(c_{i+1}^2 + c_{i+1}c_i + c_{i+1}c_{i-1} + c_i^2 + c_ic_{i-1} + c_{i-1}^2)$$
$$+ 4\mu(c_{i+1}^2 - c_{i+1}c_i - 2c_i^2 + c_ic_{i-1} + c_{i-1}^2) + 2\mu^2(- c_{i+1} + 4c_i - 3c_{i-1})]$$

where

$$\mu = \Sigma_{i=1}^1\ H_i(c_{i+1} - c_{i-1})(c_{i+1} + c_i + c_{i-1})/6.$$

Using

$$c_0 = a\ ,$$
$$c_1 = c\ ,$$
$$c_2 = b\ ,$$
$$H_1 = C\ ,$$

and the above formulas for a tetragonal probability distribution with

$$d = c\ ,$$
$$D = C\ ,$$
$$C = 2/(b - a),$$

we obtain the same formulas in the following forms:

$$\mu = H_1(c_2 - c_0)(c_2 + c_1 + c_0)/6,$$
$$\mu = C(b - a)(a + b + c)/6,$$
$$\mu = (a + b + c)/3,$$

as well as

$$\sigma^2 = 1/12\ H_1[(c_2 - c_0)(c_2^2 + c_2c_1 + c_2c_0 + c_1^2 + c_1c_0 + c_0^2)$$
$$+ 4\mu(c_2^2 - c_2c_1 - 2c_1^2 + c_1c_0 + c_0^2) + 2\mu^2(- c_2 + 4c_1 - 3c_0)]$$

and hence

$$\sigma^2 = C[(b - a)(b^2 + bc + ba + c^2 + ca + a^2)$$
$$+ 4\mu(b^2 - bc - 2c^2 + ca + a^2) + 2\mu^2(- b + 4c - 3a)]/12,$$
$$\sigma^2 = C\{(b - a)(a^2 + b^2 + c^2 + ab + ac + bc)$$
$$+ 2\mu[2(a^2 + ac - 2c^2 - bc + b^2) + \mu(- 3a - b + 4c)]\}/12.$$

Substituting

$$\mu = (a + b + c)/3,$$

we obtain

$$\sigma^2 = C\{9(b - a)(a^2 + b^2 + c^2 + ab + ac + bc)$$
$$+ 2(a + b + c)[6(a^2 + ac - 2c^2 - bc + b^2) + (a + b + c)(- 3a - b + 4c)]\}/108$$
$$= C[9(b - a)(a^2 + b^2 + c^2 + ab + ac + bc)$$
$$+ 2(a + b + c)(6a^2 + 6ac - 12c^2 - 6bc + 6b^2 - 3a^2 - 3ab - 3ac - ab - b^2 - bc + 4ac + 4bc + 4c^2)]/108$$
$$= C[(9b - 9a)(a^2 + b^2 + c^2 + ab + ac + bc)$$
$$+ (2a + 2b + 2c)(3a^2 - 4ab + 5b^2 + 7ac - 8c^2 - 3bc)]/108$$

$$= C(9a^2b - 9a^3 + 9b^3 - 9ab^2 + 9bc^2 - 9ac^2 + 9ab^2 - 9a^2b + 9abc - 9ac^2 + 9b^2c - 9abc$$
$$+ 6a^3 + 6a^2b + 6a^2c - 8a^2b - 8ab^2 - 8abc + 10ab^2 + 10b^3 + 10b^2c$$
$$+ 14a^2c + 14abc + 14ac^2 - 16ac^2 - 16bc^2 - 16c^3 - 6abc - 6b^2c - 6bc^2)/108$$
$$= C(- 3a^3 - 2a^2b + 20a^2c + 2ab^2 - 20ac^2 + 19b^3 + 13b^2c - 13bc^2 - 16c^3)/108$$

Finally

Nota bene: Similarly, we can also determine further initial and central moments etc. [Cramér], e.g. skewness

$$\gamma_1 = E[(X - \mu)^3/\sigma^3]$$

and excess

$$\gamma_2 = E[(X - \mu)^4/\sigma^4] - 3.$$

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